Module: 8
Lecture: 1

Energy dissipated by damping

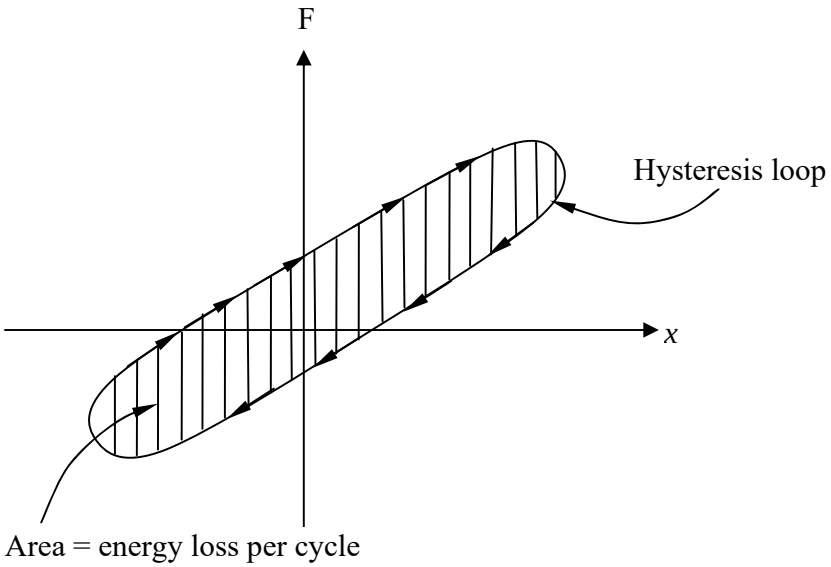
Usually damping is present in all oscillatory system. Its effect is to remove energy from the system. Energy in a vibrating system is either dissipated into heat sound or radiated away.

In vibration analysis, we are generally concerned with damping in terms of system response. The loss of energy from the oscillatory system results in the decay of amplitude of free vibration. In steady state force vibration, the loss of energy is balanced by the energy which is supplied by the excitation.

A vibrating system may encounter many different types of damping forces, from internal molecular friction to sliding friction and fluid resistance. Generally their mathematical description is quite complicated and not suitable for vibration analysis. Thus simplified damping models have been developed that in many cases are not found to be adequate in evaluating the system response.

Energy dissipated is usually determined under the condition of cyclic oscillation. Depending on the type of damping present, the force-displacement relationship when plotted may differ greatly. In all cases, however, the force-displacement curve will enclose an area, referred to as the hysteresis loop that is proportional to the energy loss per cycle.

The energy lost per cycle due to a damping force F_d is computed from the general



equation

$$W_d = \oint F_d\, dx \qquad \dot{X} = \frac{dX}{dt} \Rightarrow dX = \dot{X}dt$$

$$F_d = c\dot{X}$$

$$W_d = \oint c\dot{X}\left(\dot{X}\right)dt$$

$$= \oint c\dot{X}^2\, dt$$

We consider in this section the simplest case of energy dissipation, that of a spring-mass system with viscous damping. The damping force in this case is $F_d = c\dot{X}$. With the steady state displacement and velocity

$$x = X\sin(\omega t - \phi)$$

$$\dot{x} = \omega X\cos(\omega t - \phi)$$

$$\ddot{x} = -\omega^2 X\sin(\omega t - \phi)$$

the energy dissipated per cycle becomes

$$W_d = \oint c\dot{X}\, dx = \oint c\dot{X}^2\, dt$$

$$= c\omega^2 X^2 \int_0^{\frac{2\pi}{\omega}} \cos^2(\omega t - \phi)\, dt$$

$$= c\omega^2 X^2\left(\frac{\pi}{\omega}\right)$$

$$= \pi c\omega X^2$$

In a similar line work done by spring and inertia forces can be written as follows:

$$W_s = \oint F_s\, dx \qquad F_s = kx$$

$$= \oint k\, x\dot{x}\, dt$$

$$= \oint k\, X^2\omega\sin(\omega t - \phi)\cos(\omega t - \phi)dt$$

$$= 0$$

$$W_I = \oint F_I\, dx \qquad F_I = m\ddot{x}$$

$$= \oint m\ddot{x}\dot{x}\, dt = \oint m\left\{-\omega^2 X\sin(\omega t - \phi)\right\}\left\{\omega X\cos(\omega t - \phi)\right\}$$

$$= -\omega^3 m X^2 \oint \sin(\omega t - \phi)\cos(\omega t - \phi)\, dt = 0$$

The energy dissipated per cycle by the damping force can be represented graphically as follows. Writing the velocity in the form

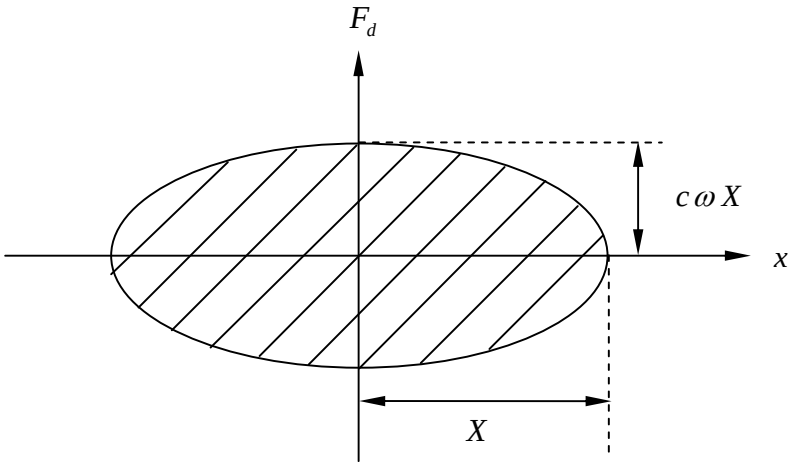
$$\begin{aligned}\dot{x} &= \omega X \cos(\omega t - \varphi) \\ &= \pm \omega X \sqrt{1 - \sin^2(\omega t - \varphi)} \\ &= \pm \omega \sqrt{X^2 - x^2(t)}\end{aligned}$$

The damping force becomes

$$F_d = c\dot{x} = \pm c\omega \sqrt{X^2 - x^2}$$

Rearranging the above equation to

$$\left(\frac{F_d}{c\omega X}\right)^2 + \left(\frac{x}{X}\right)^2 = 1$$

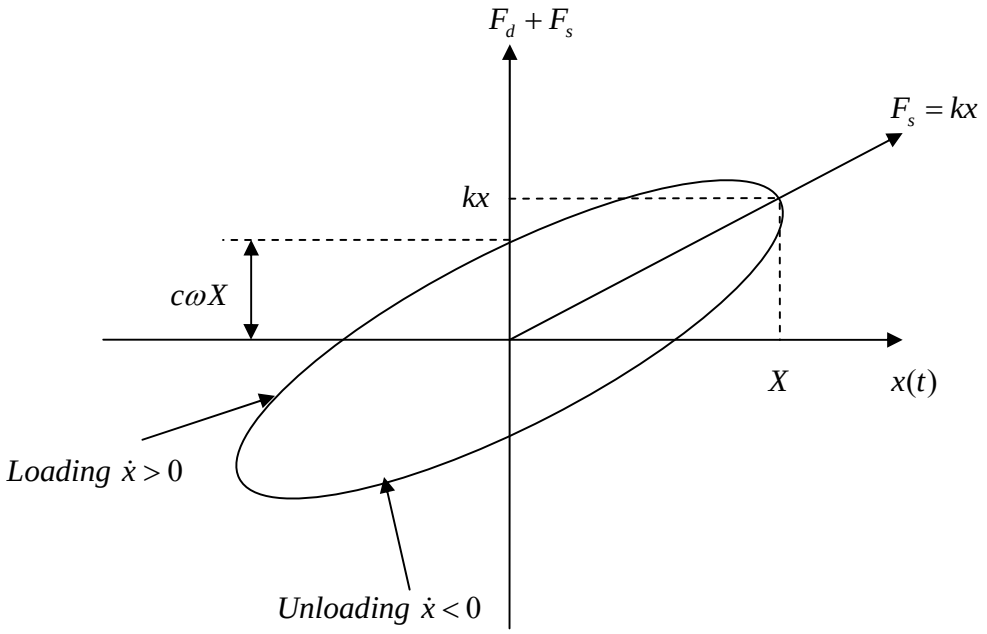


$$\text{Area} = \pi(c\omega X)(X) = \pi c\omega X^2$$

We recognize it as that of an ellipse with F_d and x plotted along the vertical and horizontal axis, as shown above. It is of interest to examine the total (elastic & damping) resisting force that is measured in an experiment.

$$\begin{aligned}F_d + F_s &= c\dot{x} + kx \\ &= kx + c\omega \sqrt{X^2 - x^2}\end{aligned}$$

A plot of $F_s + F_d$ against $x$is the ellipse. The energy dissipated by damping is still the area enclosed by the ellipse.



$F_s = k\,x$, is zero.

Damping properties of materials are listed in many different ways depending on the technical areas to which they are applied. Of these we list two relative energy units that have wide usage. First of these is specific damping capacity, defined as the energy loss per cycle w_d divided by the pick potential energy *U*.

$$\text{Specific damping capacity} = \frac{E_d}{E_{so}}$$

Where $E_{so} = \frac{1}{2} k X^2$

$$\therefore \text{ Specific damping capacity} = \frac{\pi c \omega X^2}{\frac{1}{2} k X^2} = \frac{2\pi c \omega}{\mathrm{k}}$$

The second quantity is the loss coefficient or specific damping factor, is define as the ratio of damping energy loss per radian $w_d / 2\pi$ divided by the peak potential or strain energy *U*.

$$\eta = \frac{w_d}{2\pi U} = \frac{c\omega}{k}$$

For the case of linear damping where the energy loss is proportional to the square of strain or amplitude, the hysteresis curve is an ellipse. When the damping loss is not quadratic function of the strain or amplitude, the hysteresis curve is no longer an ellipse.

Structural damping

When materials are cyclically stressed, energy is dissipated internally within the material itself. Experiments by several investigators indicate that for most structural metals, such as steel, aluminium, the energy dissipated per cycle is independent of the frequency over a wide frequency range and proportional to the square of the amplitude of vibration. Internal damping fitting this classification is called solid damping or structural damping.

Energy dissipated by structural damping may be written as

$$w_d = \alpha X^2$$

Where α is constant with units of force / displacement. Using the concept of equivalent viscous damping gives

$$\pi C_{eq} \omega X^2 = \alpha X^2$$

$$or, \quad C_{eq} = \frac{\alpha}{\pi\omega}$$

Substituting C_{eq} for c, the differential equation of motion for a system with structural damping may be written as

$$m\ddot{x} + \left(\frac{\alpha}{\pi\omega}\right)\dot{x} + kx = F(t)$$

Complex stiffness: In the calculation of the flutter speed of air plane wings and tail surfaces, the concept of complex stiffness is used. It is arrived at by assuming the oscillations to be harmonic, which enables to be written as

$$m\ddot{x} + \left(k + i\frac{\alpha}{\pi}\right)x = F_0 e^{i\omega t}$$

Defining $\gamma = \frac{\alpha}{k\pi}$, the equation becomes

$$m\ddot{x} + k(1 + i\gamma)x = F_0 e^{i\omega t}$$

The quantity $k(1 + i\gamma)$ is called complex stiffness and γ is the structural damping factor.

With the solution $x = X e^{i\omega t}$, the steady state amplitude becomes

$$X = \frac{F_0}{(k - m\omega^2) + i\gamma k}$$

The amplitude of resonance is then

$$|X| = \frac{F_0}{\gamma k}$$

Comparing this with the resonance amplitude of a system with viscous damping,

$$|X| = \frac{F_0}{2\zeta k} \qquad \gamma = 2\zeta$$

We conclude that with equal amplitudes at resonance, the structural damping factor is equal to twice the viscous damping factor.