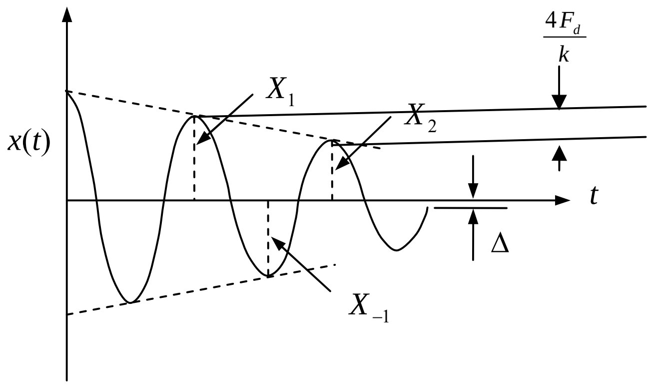
Module: 4
Lecture: 3

Damped Single-degree of Freedom System
Free Vibration of Damped System(Contd.)

Coulomb Damping

Coulomb Damping results from the sliding of two dry surfaces. The damping force is equal to the product of the normal forces and the coefficient of friction μ and is assumed to be independent of the velocity, once the motion is initiated. Since the sign of the damping force is opposite to that of velocity, the differential equation of motion for each sign is valid only for half cycle intervals.



To determine the decay of amplitude, we resort to the work-energy principle of equating the work done to the change in kinetic energy. Choosing a half-cycle staring at the extreme position with velocity equal to zero and the amplitude equal to X_1, the change in kinetic energy is zero and the work done on m is also zero.

$$\tfrac{1}{2}k\left(X_1^2 - X_{-1}^2\right) - F_d\left(X_1 + X_{-1}\right) = 0$$

$$\text{or,}\quad \tfrac{1}{2}k\left(X_1 - X_{-1}\right) - F_d = 0 \Rightarrow X_1 - X_{-1} = \frac{2F_d}{K}$$

Where X_{-1} is the amplitude after half-cycle.

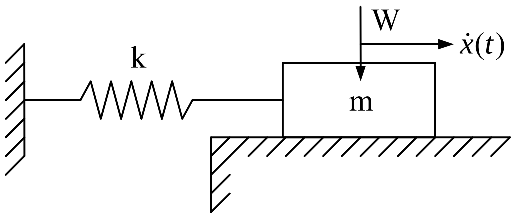
Repeating this procedure for the next half-cycle, a further decrease in amplitude of $2F_d/K$ will be found. So that decay in amplitude per cycle is a constant and equal to

$$X_1 - X_2 = \frac{4F_d}{k}$$

The motion will cease, however, when the amplitude is less than Δ, at which position the spring force is insufficient to over come static friction force, which is generally greater than the kinetic friction force.

Coulomb Damping: Dry Friction

Coulomb damping arises when the oscillating bodies are on the dry surface. For motion to begin, there must be a force acting upon the body that over comes the resistance to motion caused by friction.



Denoting by F_d the magnitude of the damping force, where $F_d = \mu_k{}^w$, the equation of motion can be written in the form

$$m\ddot{x} + F_d \operatorname{sgn}(\dot{x}) + k x = 0$$

Where $\operatorname{sgn}()$ denotes sign of and represents a function having the value $+1$ if its arguments $\dot{x}$ is positive and the value -1 if its arguments is negative. Mathematically

$$\operatorname{sgn}(\dot{x}) = \frac{\dot{x}}{|\dot{x}|}$$

$\therefore$ We can write

$$m\ddot{x} + k x = -F_d \qquad for \;\; \dot{x} > 0$$
$$m\ddot{x} + k x = F_d \qquad for \;\; \dot{x} < 0$$

Take $x(0) = x_0$, where the initial displacement x_0 sufficiently large that the restoring force in the spring exceed the static friction force. Because in the ensuing motion the velocity is negative, we must solve second equation first and the solution can be obtained as

a) Complementary solution and

b) Particular solution

$$(a)\ \ddot{x} + \omega_n^2 = 0 \rightarrow x(t) = \frac{v_1}{\omega_n}\sin\omega_n t + x_1\cos\omega_n t$$

Note: Solution of undamped SDOF with initial displacement (x_1) and velocity (v_1)

where $\omega_n^2 = \dfrac{k}{m}$

$(b)\ \ddot{x} + \omega_n^2 x = \omega_n^2 f_d$

In which $f_d = F_d/k$ represent an equivalent displacement. Now we can solve the above equation with initial conditions as $x(0) = x_0$ & $\dot{x}(0) = 0$ and the solution is

$$x(t) = (x_0 - f_d)\cos\omega_n t + f_d$$

Differentiating w.r.t.t we get,

$$\dot{x}(t) = -\omega_n(x_0 - f_d)\sin\omega_n t$$

the condition $\dot{x}(t) = 0$ leads to $t_1 = \dfrac{\pi}{\omega_n}$ at which the displacement is $x(t_1) = -x_0 + 2f_d$. If $x(t_1)$ is sufficiently large in magnitude to overcome the static friction, then the mass acquires a positive velocity, so that the motion must satisfy the equation

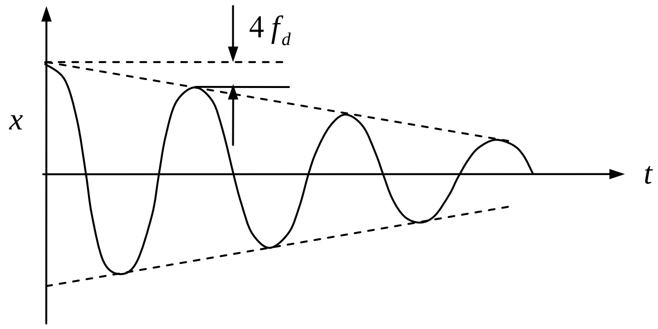
$$\ddot{x} + \omega_n^2 x = -\omega_n^2 f_d$$

Where $x(t)$ is subject to initial condition $x(t_1) = -x_0 + 2f_d$, $\dot{x}(t_1) = 0$ .The solution is

$$x(t) = (x_0 - 3f_d)\cos\omega_n t - f_d$$

Solution is valid in the interval $t_1 \le t \le t_2$ where t_2 is the next value of time in which the velocity reduces to zero. This value is $t_2 = \dfrac{2\pi}{\omega_n}$ at which time the velocity is ready to reverse direction once again, this time from right to left. The displacement at $t = t_2$ is

$$x(t_2) = x_0 - 4f_d$$



Complementary Solution

$$\ddot{x} + \omega_n^2 x = 0$$

$$x(t) = x_1 \cos\omega_n t + \frac{v_1}{\omega_n}\sin\omega_n t$$

Particular Solution

$$\ddot{x} + \omega_n^2 x = \omega_n^2 f_d$$

$$\Rightarrow \left(D^2 + \omega_n^2\right)x = \omega_n^2 f_d$$

$$\begin{aligned}
\Rightarrow x \quad &= \frac{\omega_n^2 f_d}{D^2 + \omega_n^2} \\
&= \frac{\omega_n^2 f_d}{\omega_n^2\left(1 + \dfrac{D^2}{\omega_n^2}\right)} \\
&= \left(1 + \frac{D^2}{\omega_n^2}\right)^{-1} f_d \\
&\approx \left(1 - \frac{D^2}{\omega_n^2}\right) f_d \\
&= f_d
\end{aligned}$$

$\therefore$ Total Solution:

$$x(t) = x_1 \cos\omega_n t + \frac{v_1}{\omega_n}\sin\omega_n t + f_d$$

$$\dot{x}(t) = -\omega_n x_1 \sin\omega_n t + v_1 \cos\omega_n t$$

$t = 0$ $$x_0 = x_1 + f_d \Rightarrow x_1 = (x_0 - f_d)$$

$$x(t) = (x_0 - f_d)\cos\omega_n t + f_d$$