

Module: 4

Lecture:2

Damped Single-degree of Freedom System Free Vibration of Damped System(Contd.)

Critically damped motion $[\zeta = 1.0]$ Case - III

We obtain a double root $s_1 = s_2 = -\omega_n$

The correct general solution is

$$x = (A + Bt)e^{-\omega_n t}$$

Initial conditions $x(0), \dot{x}(0)$ at $t = 0$

$$\dot{x} = -\omega_n A e^{-\omega_n t} + B e^{-\omega_n t} - B\omega_n t e^{-\omega_n t}$$

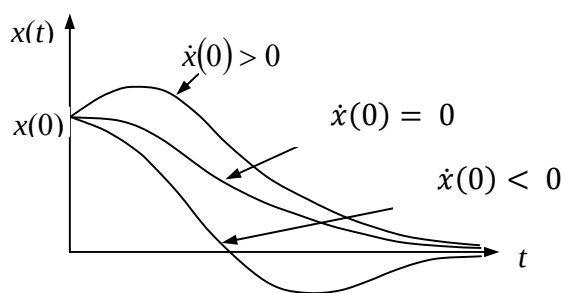
$$\text{at } t = 0 \quad x(0) = A$$

$$\text{at } t = 0 \quad \dot{x}(0) = -\omega_n A + B$$

$$\Rightarrow B = \dot{x}(0) + \omega_n A$$

$$= \dot{x}(0) + \omega_n x(0)$$

$$\therefore x = [x(0) + \{\dot{x}(0) + \omega_n x(0)\}t]e^{-\omega_n t}$$



Oscillatory Motion [$\zeta > 1.0$] Case I

$$x = Ae^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + Be^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

$$\dot{x} = A(-\zeta + \sqrt{\zeta^2 - 1})\omega_n e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + B(-\zeta - \sqrt{\zeta^2 - 1})\omega_n e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

B . C. at $t = 0$ $x = x(0)$, $\dot{x} = \dot{x}(0)$

B . C (1) $\rightarrow x(0) = A + B \rightarrow A = x(0) - B$

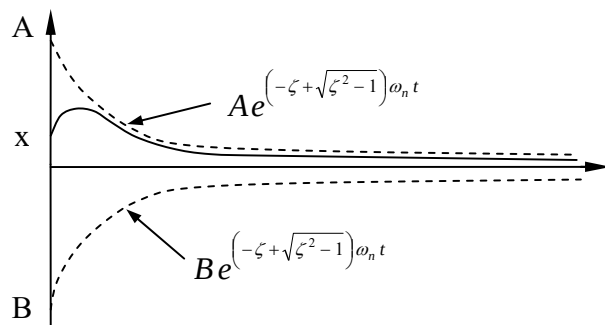
B . C. (2) $\rightarrow \dot{x}(0) = A(-\zeta + \sqrt{\zeta^2 - 1})\omega_n + B(-\zeta - \sqrt{\zeta^2 - 1})\omega_n$

$$\begin{aligned} \dot{x}(0) &= \{x(0) - B\}(-\zeta + \sqrt{\zeta^2 - 1})\omega_n + B(-\zeta - \sqrt{\zeta^2 - 1})\omega_n \\ \Rightarrow \dot{x}(0) + x(0)(-\zeta - \sqrt{\zeta^2 - 1})\omega_n &= B\{-\zeta + \sqrt{\zeta^2 - 1} - \zeta - \sqrt{\zeta^2 - 1}\} \\ B &= \frac{-\dot{x}(0) - x(0)(-\zeta - \sqrt{\zeta^2 - 1})\omega_n}{2\sqrt{\zeta^2 - 1}\omega_n} \end{aligned}$$

$$A = x(0) - B$$

$$\begin{aligned} &= x(0) - \frac{-\dot{x}(0) - x(0)(-\zeta - \sqrt{\zeta^2 - 1})\omega_n}{2\sqrt{\zeta^2 - 1}\omega_n} \\ &= \frac{\dot{x}(0) + x(0)(\zeta + \sqrt{\zeta^2 - 1})\omega_n}{2\omega_n\sqrt{\zeta^2 - 1}} \end{aligned}$$

$$x = Ae^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + Be^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \rightarrow \text{Response equation}$$



Oscillatory motion ($\zeta > 1.0$)

Special Case (From General damped solution)

Free undamped vibration

$$c = 0$$

$$x = A_1 \sin \omega_n t + A_2 \cos \omega_n t$$

Where $\omega_n = \sqrt{k/m} \rightarrow$ natural frequency

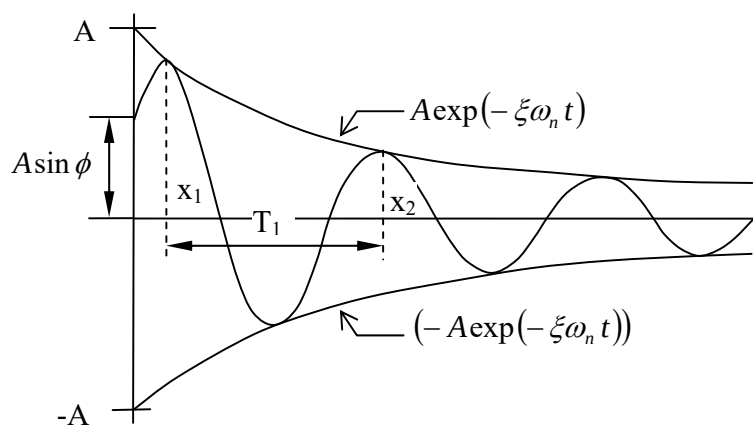
$$x = A \sin(\omega_n t + \phi)$$

To determine A_1 , A_2 and A we have to apply the initial condition at $t=0$.

at $t = 0$ $x = x_0$, $\dot{x} = v_0$

$$x(t) = x_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t$$

Logarithmic Decrement from Damped free vibration



Height of two consecutive picks x_1 , x_2 . We can define a quantity called Logarithmic Decrement

$$\begin{aligned}\delta &= \ln \frac{x_1}{x_2} = \ln \frac{Ae^{-\xi \omega_n t} \sin(\omega_d t + \phi)}{Ae^{-\xi \omega_n (t+T)} \sin[\omega_d (t+T) + \phi]} \\ &= \ln \frac{e^{-\xi \omega_n t} \sin(\omega_d t + \phi)}{e^{-\xi \omega_n (t+T)} \sin(\omega_d t + \phi)} \quad T = \frac{2\pi}{\omega_d}\end{aligned}$$

$$\sin\left[\omega_d \left(t + \frac{2\pi}{\omega_d}\right) + \phi\right] = \sin(\omega_d t + 2\pi + \phi) = \sin(\omega_d t + \phi)$$

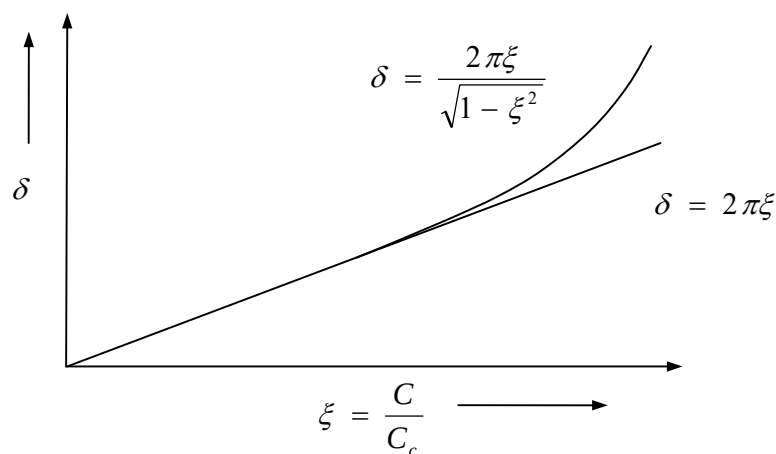
$$\begin{aligned}\delta &= \xi \omega_n T = \xi \omega_n \frac{2\pi}{\omega_d} \\ &= \xi \omega_n \frac{2\pi}{\omega_n \sqrt{1-\xi^2}} \quad \xi \rightarrow \text{very small}\end{aligned}$$

$$\delta = 2\pi\xi \Rightarrow \xi = \frac{\delta}{2\pi}$$

The damping factor ξ can also be determined by measuring two displacements separated by any number of complete cycles. Letting x_1 and x_{j+1} be the amplitudes corresponding to the times t_1 and $t_{j+1} = t_1 + jT$, where j is an integer, we conclude that

$$\frac{x_1}{x_{j+1}} = \frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdot \frac{x_3}{x_4} \cdots \frac{x_j}{x_{j+1}} = e^{(\xi \omega_n T) j}$$

$$\delta = \frac{1}{j} \ln \frac{x_1}{x_{j+1}} = \frac{1}{j} \ln (e^{(\xi \omega_n T) j}) = \xi \omega_n T$$



Graphical representation of linear and non-linear logarithmic decrement curves

Summary of Expressions

$$\lambda_{1,2} = \left(-\xi \pm \sqrt{\xi^2 - 1} \right) \omega_n$$

when $\xi > 1 \rightarrow \lambda_1 \text{ \& \; } \lambda_2$ are real

$$x = A_1 \exp\left(-\xi + \sqrt{\xi^2 - 1}\right) \omega_n t + A_2 \exp\left(-\xi - \sqrt{\xi^2 - 1}\right) \omega_n t$$

when $\xi = 1 \quad \lambda_{1,2} = -\omega_n$

$$x = (A_1 + A_2 t) e^{-\omega_n t}$$

when $\xi < 1$, $\lambda_{1,2}$ are complex

$$x = A_1 \exp\left(-\xi + i\sqrt{\xi^2 - 1}\right) \omega_n t + A_2 \exp\left(-\xi - i\sqrt{\xi^2 - 1}\right) \omega_n t$$

$$\omega_d = \omega_n \sqrt{1 - \xi^2}$$

$$\delta = \ln \frac{x_1}{x_2} = \frac{2\pi\xi}{\sqrt{1 - \xi^2}}$$

For small damping, $\delta = 2\pi\xi$

When damping is very small you should take a series of peaks (2 peaks will give erroneous result)

$$\delta = \frac{1}{n} \ln \frac{x_1}{x_{n+1}}$$

n = no of peaks

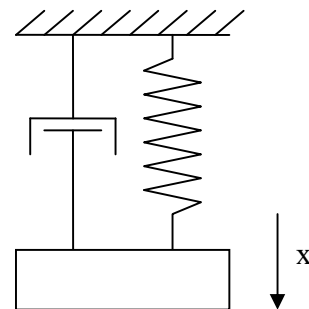
Numerical Example

Problem: Initial condition and damping conditions are given as follows

$$t = 0 ; x = 0 ; \dot{x} = v_0$$

$$\xi = 2.5, 1, 0.1$$

Find out the response



Solution:

There are three cases

- 1) Over damped
- 2) Critically damped
- 3) Under damped

Case I $\xi = 2.5$

$$x = A_1 \exp\left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n t + A_2 \exp\left(-\xi - \sqrt{\xi^2 - 1}\right)\omega_n t$$

$$x = A_1 e^{-0.209\omega_n t} + A_2 e^{-4.79\omega_n t}$$

To find A_1, A_2 use initial condition

$$\Rightarrow A_1 = -A_2 = \frac{v_0}{\omega_n}(0.218)$$

$$x = 0.218 \frac{v_0}{\omega_n} \left(e^{-0.209\omega_n t} + e^{-4.79\omega_n t} \right)$$

Case II

$$\xi = 1$$

$$x = (A_1 + A_2 t)e^{-\omega_n t}$$

Applying B. C. $\Rightarrow A_1 = 0 ; A_2 = v_0$

$$x = V_0 t e^{\omega_n t}$$

Case III

$$\xi = 0.1$$

$$x = e^{-0.1\omega_n t} (A_1 \sin 0.995\omega_n t + B \cos 0.995\omega_n t)$$

From the initial conditions

$$x = 0 \rightarrow B = 0$$