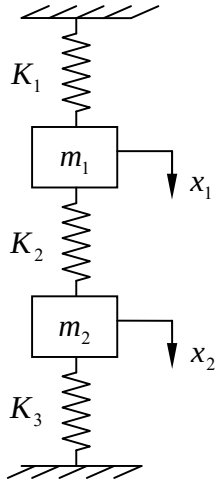
Module: 11
Lecture: 1

System with two DOF

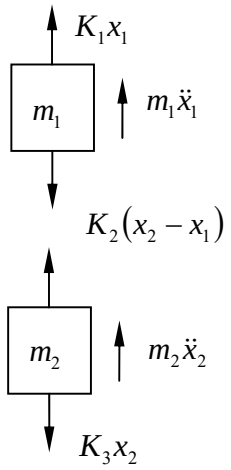
Free Undamped Vibration



Let us assume that the two degrees of freedoms are measured as x_1 and x_2 as indicated in the figure.

A two DOF system has 2 natural frequencies. When a system vibrates at any natural frequencythere exists a deflection relationship with respect to amplitudes.

Such a configuration is called normal mode. In a normal mode oscillation both m_1 and m_2 have harmonic motion of same frequency passing through simultaneously through the equilibrium position.



Considering the equilibrium of

mass $m_1 \Rightarrow m_1\ddot{x}_1 + K_1x_1 - K_2(x_2 - x_1) = 0$

or, $m_1\ddot{x}_1 + (K_1 + K_2)x_1 - K_2x_2 = 0 \rightarrow (1)$

Equilibrium of mass $m_2 \Rightarrow m_2\ddot{x}_2 + K_2(x_2 - x_1) + K_3x_2 = 0$

or, $m_2\ddot{x}_2 - K_2x_1 + (K_2 + K_3)x_2 = 0 \rightarrow (2)$

Express (1) & (2) in matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2+K_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=0$$

Assuming m_1 & m_2 oscillating at the same frequancy, but at different amplitudes.

$$\begin{aligned} & x_1 = A_1 \sin(\omega t+\alpha) \\ & x_2 = A_2 \sin(\omega t+\alpha) \rightarrow (3) \\ \Rightarrow \quad & x_1 = A_1 e^{i\omega t} \\ & x_2 = A_2 e^{i\omega t} \end{aligned} \Bigg\} \quad \text{alternative form}$$

Substituting (3) in matrix equation

$$\begin{aligned} & -m_1 A_1 \omega^2 + (K_1+K_2)A_1 - K_2 A_2 = 0 \\ & -m_2 A_2 \omega^2 + (K_2+K_3)A_2 - K_2 A_1 = 0 \end{aligned}$$

$$\Rightarrow \quad \begin{bmatrix} -\omega^2+\dfrac{K_1+K_2}{m_1} & -\dfrac{K_2}{m_1} \\ -\dfrac{K_2}{m_2} & -\omega^2+\dfrac{K_2+K_3}{m_2} \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0$$

For non-trivial solution

$$\begin{vmatrix} -\omega^2+\dfrac{K_1+K_2}{m_1} & -\dfrac{K_2}{m_1} \\ -\dfrac{K_2}{m_2} & -\omega^2+\dfrac{K_2+K_3}{m_2} \end{vmatrix}=0$$

$$\Rightarrow \qquad \omega^4 - \omega^2\left[\frac{K_1+K_2}{m_1}+\frac{K_2+K_3}{m_2}\right]+\left[\left(\frac{K_1+K_2}{m_1}\right)\left(\frac{K_2+K_3}{m_2}\right)-\frac{K_2{}^2}{m_1m_2}\right]=0$$

$$\omega_{1,2}{}^2 = \frac{1}{2}\left[\frac{K_1+K_2}{m_1}+\frac{K_2+K_3}{m_2}\right]$$

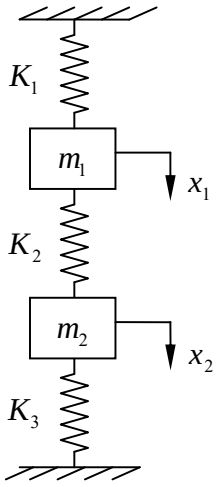
$$\pm\frac{1}{2}\left[\left\{\frac{K_1+K_2}{m_1}+\frac{K_2+K_3}{m_2}\right\}^2-4\left\{\left(\frac{K_1+K_2}{m_1}\right)\left(\frac{K_2+K_3}{m_2}\right)-\frac{K_2{}^2}{m_1m_2}\right\}\right]^{1/2}$$

$$=\frac{1}{2}\left[\frac{K_1+K_2}{m_1}+\frac{K_2+K_3}{m_2}\right]\pm\frac{1}{2}\left[\left\{\frac{K_1+K_2}{m_1}-\frac{K_2+K_3}{m_2}\right\}^2+\frac{4K_2{}^2}{m_1m_2}\right]^{1/2} \qquad \because (a-b)^2=(a+b)^2-4ab$$

$$\left(\frac{A_1}{A_2}\right)_{\omega_1{}^2}=\frac{-K_2}{m_1\omega_1{}^2-K_1-K_2}=\frac{m_2\omega_2{}^2-K_2-K_3}{-K_2}$$

$$\left(\frac{A_1}{A_2}\right)_{\omega_2{}^2}=\frac{-K_2}{m_1\omega_2{}^2-K_1-K_2}=\frac{m_2\omega_2{}^2-K_2-K_3}{-K_2}$$

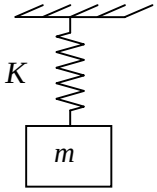
Problem 1 Find out the mode shapes of the following spring mass systems.

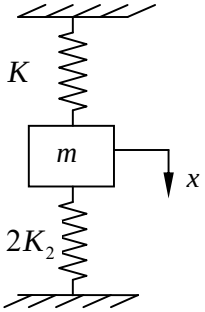


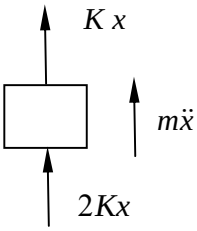
$$\omega_1^2 = \frac{K}{m} \qquad \omega_2^2 = \frac{K + 2K_2}{m}$$

$$\omega_1 \rightarrow \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

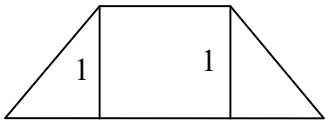
$$\omega_2 \rightarrow \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$



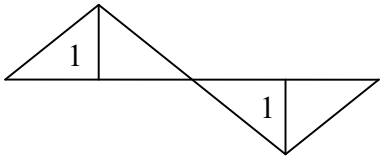




$$\left|\frac{A_1}{A_2}\right|_{\omega_1^2 = \frac{K}{m}} = 1 \quad ; \qquad \left|\frac{A_1}{A_2}\right|_{\omega_2^2 = \frac{K+2K_2}{m}} = -1$$



Two masses move in the same direction through the same distance



Two masses move through same distance but in the opposite direction.