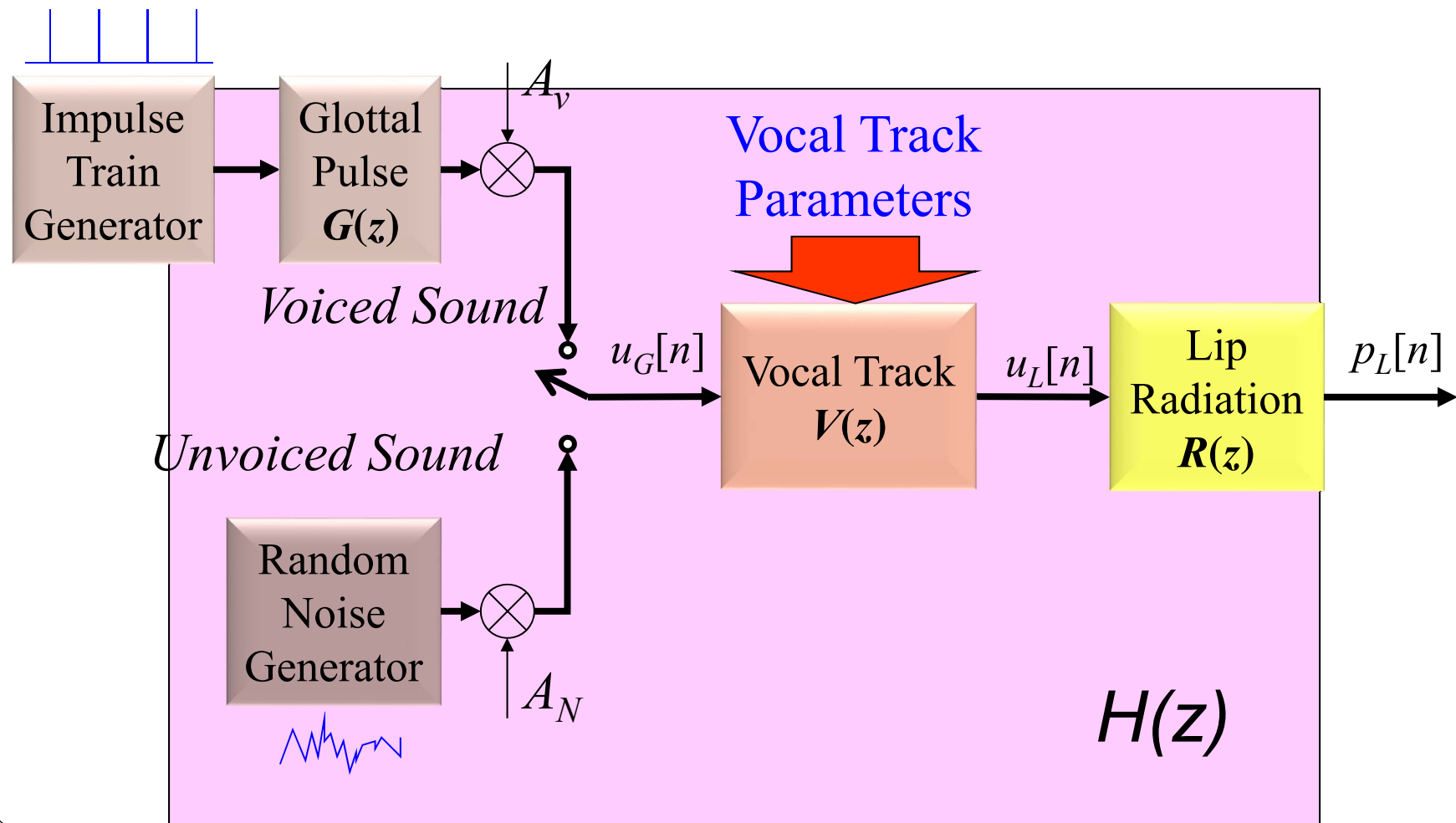


Speech Production Model

$$H(z) = G(z)V(z)R(z)$$



Speech Production Model

$$H(z) = G(z)V(z)R(z)$$

$$G(z) = \frac{1}{(1 - e^{-cT} z^{-1})^2}$$

$$V(z) = \frac{G}{\prod_{k=1}^{N/2} (1 - 2r_k \cos \theta_k z^{-1} + r_k^2 z^{-2})}$$

$$R(z) = R_0 (1 - z^{-1})$$

$$H(z) = \frac{\sigma(1 - z^{-1})}{(1 - e^{-cT} z^{-1})^2 \prod_{k=1}^{N/2} (1 - 2r_k \cos \theta_k z^{-1} + r_k^2 z^{-2})}$$

All pole modeling

$$H(z) = G(z)V(z)R(z)$$

$$G(z) = \frac{1}{(1 - e^{-cT} z^{-1})^2} \quad e^{-cT} \approx 1$$

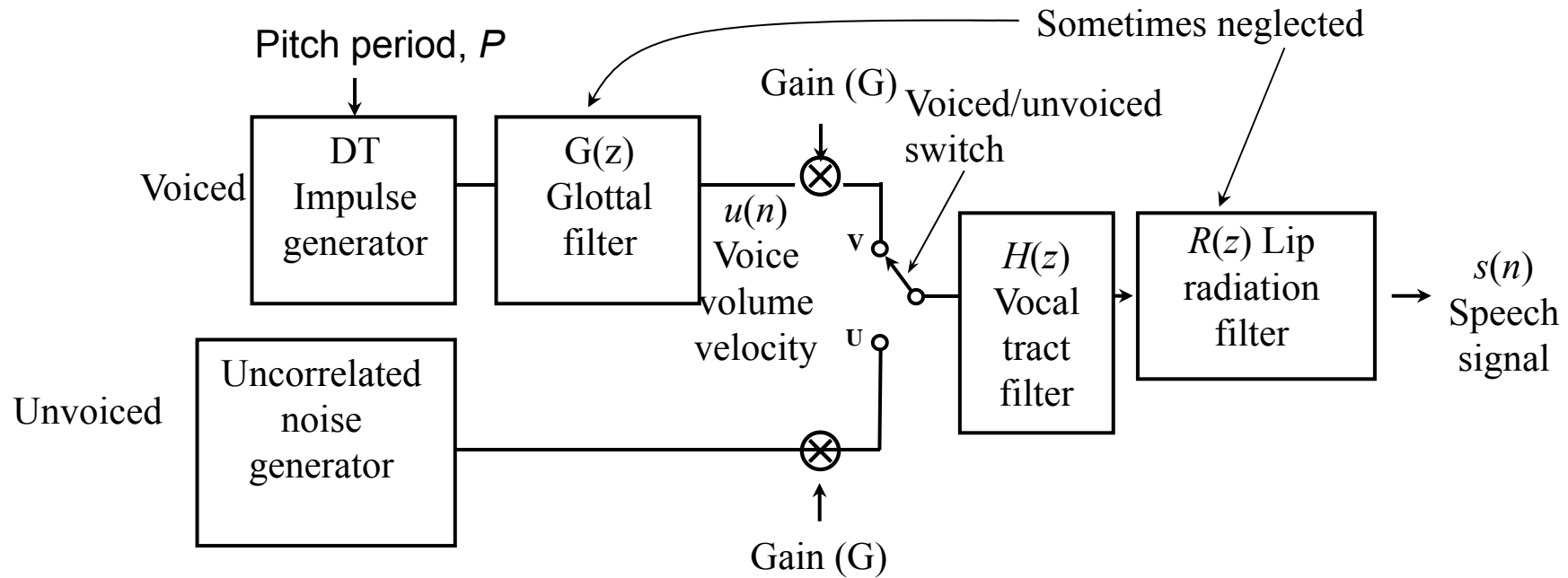
$$V(z) = \frac{G}{\prod_{k=1}^{N/2} (1 - 2r_k \cos \theta_k z^{-1} + r_k^2 z^{-2})}$$

$$R(z) = R_0(1 - z^{-1})$$

$$H(z) = \frac{\sigma}{1 - \sum_{k=1}^p a_k z^{-k}}$$

$$H(z) = \frac{GR_0(1 - z^{-1})}{(1 - e^{-cT} z^{-1})^2 \prod_{k=1}^{N/2} (1 - 2r_k \cos \theta_k z^{-1} + r_k^2 z^{-2})}$$

General Speech model:



$$H(z) = \frac{G}{1 + \sum_{k=1}^M a_k z^{-k}}$$

The LPC Model

$$H(z) = \frac{S(z)}{U(z)} = \frac{A}{1 - \sum_{k=1}^p a_k z^{-k}}$$

$$S(z)[1 - \sum_{k=1}^p a_k z^{-k}] = AU(z)$$

$$S(z) - \sum_{k=1}^p a_k S(z) z^{-k} = AU_g(z)$$

$$S(z) = \sum_{k=1}^p a_k z^{-k} S(z) + AU_g(z)$$

$$s[n] = \sum_{k=1}^p a_k s[n-k] + Au_g[n],$$

$$s[n] = \sum_{k=1}^p a_k s[n-k] \quad \text{when} \quad u_g(n) = 0$$

$$s[n] \approx a_1 s[n-1] + a_2 s[n-2] + \dots + a_p s[n-p]$$

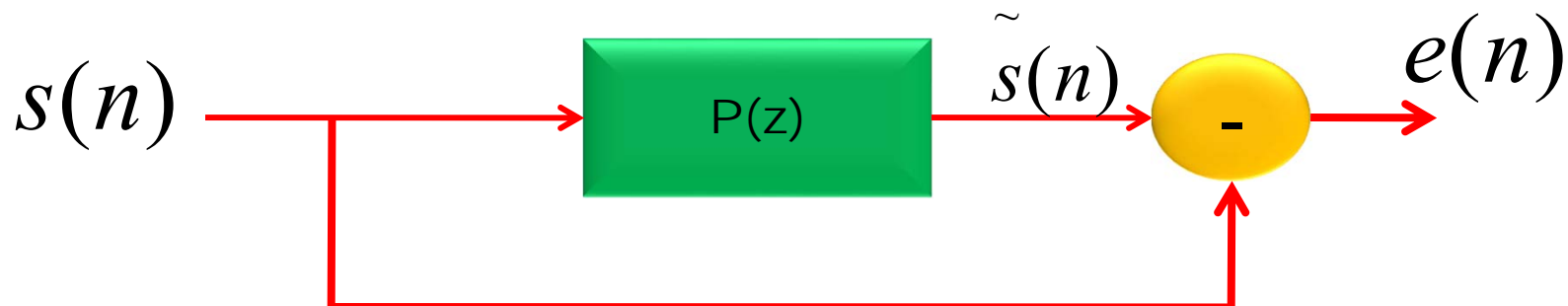
Liner predictor

$$\tilde{s}[n] = \sum_{k=1}^p \alpha_k s[n-k]$$

$$P(z) = \sum_{k=1}^p \alpha_k z^{-k} \quad \text{FIR Filter of order } P$$

The prediction error:

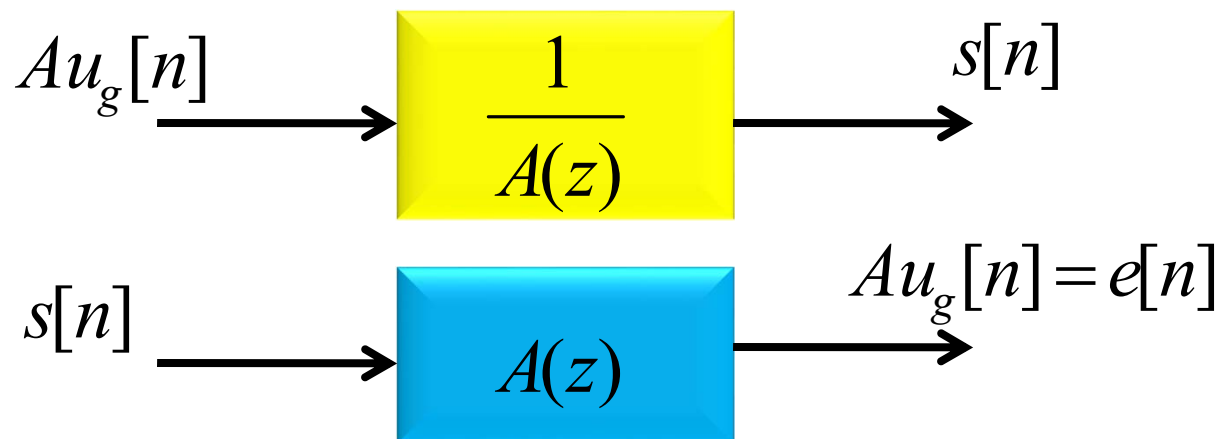
$$e[n] = s[n] - \tilde{s}[n] = s[n] - \sum_{k=1}^p \alpha_k s[n-k]$$



Error transfer function:

$$e[n] = s[n] - \tilde{s}[n] = s[n] - \sum_{k=1}^p \alpha_k s[n-k]$$

$$A(z) = \frac{E(z)}{S(z)} = 1 - \sum_{k=1}^p \alpha_k z^{-k} = 1 - P(z)$$



LP Estimation Issues



- Need to determine $\{\alpha_k\}$ directly from speech such that they give good estimates of the time-varying spectrum
- Need to estimate $\{\alpha_k\}$ from short segments of speech
- Need to minimize mean-squared prediction error over short segments of speech
- resulting $\{\alpha_k\}$ assumed to be the actual $\{a_k\}$ in the speech production model

all of this can be done efficiently, reliably, and accurately for speech

Solution for $\{\alpha_k\}$

$$e_n(m) = s_n[m] - \sum_{k=1}^p \alpha_k s_n[m-k] \quad \text{where} \quad n-m \leq m \leq n+m$$

mean squared error signal:

$$E_n = \sum_m e_n^2[m]$$

$$E_n = \sum_m \left[s_n[m] - \sum_{k=1}^p \alpha_k s_n[m-k] \right]^2$$

Can find values of α_k that minimize by setting

$$\frac{\partial E_n}{\partial \alpha_i} = 0, \quad i = 1, 2, \dots, p$$

$$\begin{aligned} \frac{\partial E_n}{\partial \alpha_i} &= \frac{\partial}{\partial \alpha_i} \sum_{-\infty}^{\infty} (s_n[m] - \sum_{k=1}^p \alpha_k s_n[m-k])^2 \\ &= 2 \sum_{-\infty}^{\infty} (s_n[m] - \sum_{k=1}^p \alpha_k s_n[m-k]) \left(-\frac{\partial}{\partial \alpha_i} \sum_{k=1}^p \alpha_k s_n[m-k] \right) \end{aligned}$$

Where

$$-s_n[m-i] = -\frac{\partial}{\partial \alpha_i} \sum_{k=1}^p \alpha_k s_n[m-k]$$

$\alpha_k s_n[m-k]$ is constant with respect to $\frac{\partial}{\partial \alpha_i}$ for $k \neq i$

$$0 = 2 \sum_{-\infty}^{\infty} (s_n[m] - \sum_{k=1}^p \alpha_k s_n[m-k]) (-s_n[m-i])$$

$$\sum_{-\infty}^{\infty} s_n[m-i] s_n[m] = \sum_{k=1}^p \alpha_k \sum_{-\infty}^{\infty} s_n[m-i] s_n[m-k] \quad 1 \leq i \leq p \quad (1)$$

let

$$\phi_n[i, k] = \sum_m s_n[m-i] s_n[m-k] \quad 1 \leq i \leq p$$

then

$$\phi_n[i, 0] = \sum_{k=1}^p \alpha_k \phi_n[i, k] \quad i = 1, 2, \dots, p \quad (2)$$

leading to a set of p equations in p unknowns that can be solved in an efficient manner for the $\{\alpha_k\}$

$$\begin{bmatrix} \phi_n[1,1] & \phi_n[1,2] & \phi_n[1,3] & \cdots & \phi_n[1,p] \\ \phi_n[2,1] & \phi_n[2,2] & \phi_n[2,3] & \cdots & \phi_n[2,p] \\ \phi_n[3,1] & \phi_n[3,2] & \phi_n[3,3] & \cdots & \phi_n[3,p] \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_n[p,1] & \phi_n[p,2] & \phi_n[p,3] & \cdots & \phi_n[p,p] \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ . \\ \alpha_p \end{bmatrix} = \begin{bmatrix} \phi_n[1,0] \\ \phi_n[2,0] \\ \phi_n[3,0] \\ \vdots \\ \phi_n[p,0] \end{bmatrix}$$

The resulting covariance matrix is symmetric, but not Toeplitz, and can be solved efficiently by a set of techniques called Cholesky decomposition

$$E_n = \sum_{m=-\infty}^{\infty} e_n^2[m]$$

$$E_n = \sum_{m=-\infty}^{\infty} [s_n[m] - \sum_{k=1}^p \alpha_k s_n[m-k]]^2$$

$$= \sum_{m=-\infty}^{\infty} s_n^2[m] - 2 \sum_{m=-\infty}^{\infty} s_n^2[m] \sum_{k=1}^p \alpha_k s_n[m-k]$$

$$+ \sum_{m=-\infty}^{\infty} \sum_{k=1}^p \alpha_k s_n[m-k] \sum_{l=1}^p \alpha_l s_n[m-l]$$

$$= \sum_{m=-\infty}^{\infty} s_n^2[m] - \sum_{k=1}^p \alpha_k \sum_{m=-\infty}^{\infty} s_n[m-k] s_n[m]$$

$$= \varphi_n[0,0] - \sum_{k=1}^p \alpha_k \varphi_n[0,k]$$

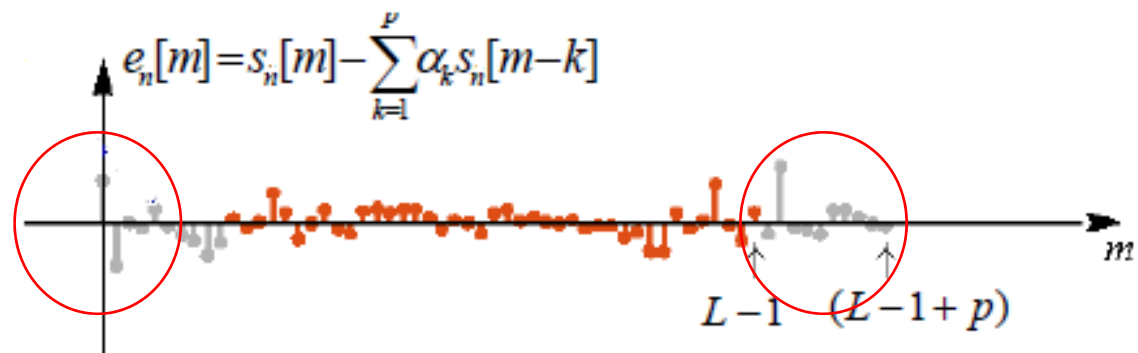
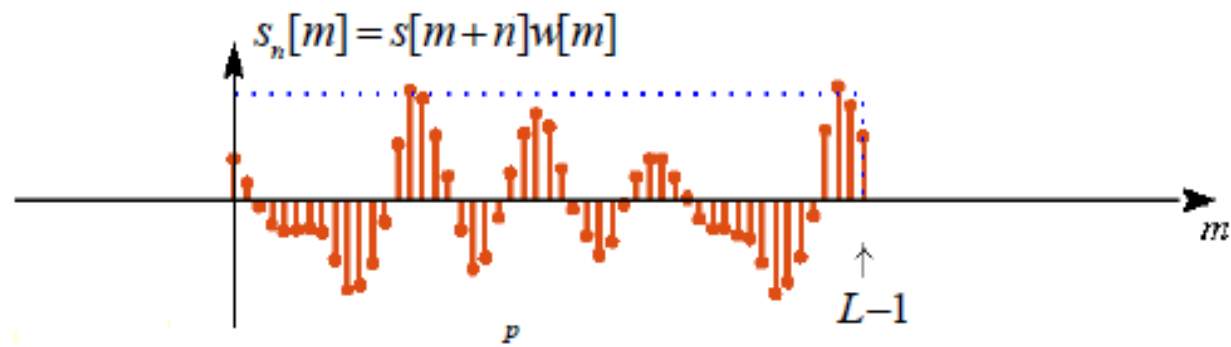
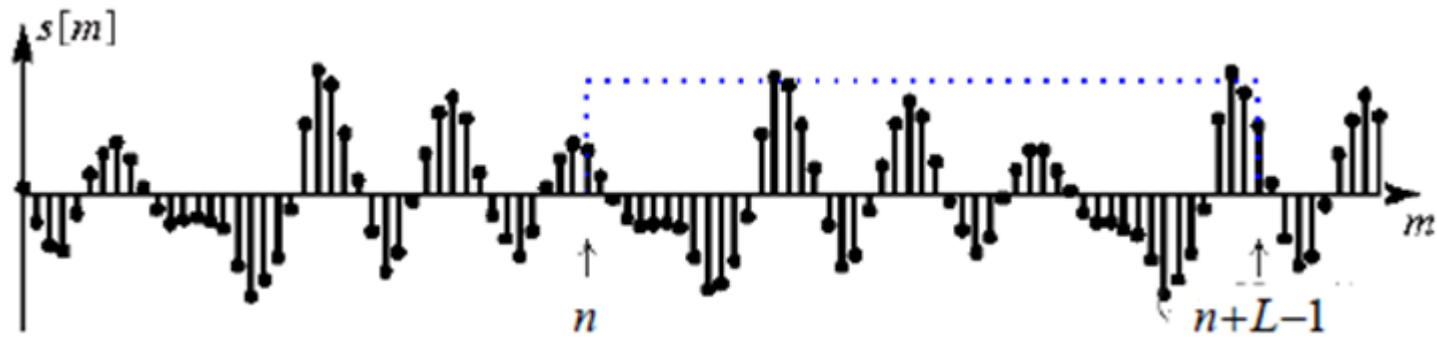
Autocorrelation Method

Let $s_n[m]$ exists for $0 \leq m \leq L - 1$ and is exactly zero every where else (i.e., window of length samples)

$$s_n[m] = s[m]w[m]$$

Where $w[m]$ is a finite length window of L samples





$$e_n(m) = s_n[m] - \sum_{k=1}^p \alpha_k s_n[m-k]$$

Is only non zero over the interval $0 \leq m \leq L-1+p$

$$E_n = \sum_m e_n^2[m] = \sum_{m=0}^{L-1+p} e_n^2[m]$$

$$\phi_n[i, k] = \sum_m S_n[m-i] S_n[m-k] \quad 1 \leq i \leq p, \quad 1 \leq k \leq p$$

since

$$S_n[m] = 0 \quad \text{outside} \quad \text{the} \quad 0 \leq m \leq L-1$$

$$\phi_n[i, k] = \sum_{m=0}^{L-1+p} S_n[m-i] S_n[m-k] \quad 1 \leq i \leq p, \quad 0 \leq k \leq p$$

$$\phi_n[i, k] = \sum_{m=0}^{L-1+(i-k)} S_n[m] S_n[m+i-k] \quad 1 \leq i \leq p, \quad 0 \leq k \leq p$$



There are $L-|i-k|$ non-zero terms in the computation of $\phi_n[i, k]$ for each value of i, k

Then $\phi_n[i, k] = R_n[i - k]$ short-time autocorrelation

$$R_n[k] = \sum_{m=0}^{L-1+k} S_n[m] S_n[m + k]$$

From equation (2)

$$\sum_{k=1}^p \alpha_k \phi_n[i, k] = \phi_n[i, 0] \quad 1 \leq i \leq p$$

$$\sum_{k=1}^p \alpha_k R_n[i - k] = R_n[i] \quad 1 \leq i \leq p$$

Minimum mean-squared prediction error can be written as

$$\begin{aligned} E_n &= \phi_n[0, 0] - \sum_{k=1}^p \alpha_k \phi_n[0, k] \\ &= R_n[0] - \sum_{k=1}^p \alpha_k R_n[k] \end{aligned}$$



$$\begin{bmatrix} R_n[0] & R_n[1] & R_n[2] & \cdots & R_n[p-1] \\ R_n[1] & R_n[0] & R_n[1] & \cdots & R_n[p-2] \\ R_n[2] & R_n[1] & R_n[0] & \cdots & R_n[p-3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_n[p-1] & R_n[p-2] & R_n[p-3] & \cdots & R_n[0] \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} R_n[1] \\ R_n[2] \\ R_n[3] \\ \vdots \\ R_n[p] \end{bmatrix}$$

$$\mathfrak{R} \alpha = r$$

$$\alpha = \mathfrak{R}^{-1} r$$

\mathfrak{R} is a $p \times p$ Toeplitz Matrix \Rightarrow symmetric with all diagonal elements equal

matrix equation solved using Levinson or Durbin method



The set of optimum predictor coefficients satisfy

$$\sum_{k=1}^p \alpha_k R_n[i-k] - R_n[i] = 0 \quad 1 \leq i \leq p$$

with minimum mean-squared prediction error of

$$R_n[0] - \sum_{k=1}^p \alpha_k R_n[k] = E^{(p)}$$



$$\begin{bmatrix} R_n[0] & R_n[1] & R_n[2] & \cdots & R_n[p] \\ R_n[1] & R_n[0] & R_n[1] & \cdots & R_n[p-1] \\ R_n[2] & R_n[1] & R_n[0] & \cdots & R_n[p-2] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_n[p-1] & R_n[p-2] & R_n[p-3] & \cdots & R_n[0] \end{bmatrix} \begin{bmatrix} 1 \\ -\alpha_1^{(p)} \\ -\alpha_2^{(p)} \\ \vdots \\ -\alpha_p^{(p)} \end{bmatrix} = \begin{bmatrix} E_n^p \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Expanded matrix is still Toeplitz and It can be solved iteratively by incorporating new correlation value at each iteration and solving for next higher order predictor in terms of new correlation value and previous predictor

i^{th} order solution can be derived from $(i-1)^{st}$ order solution

given $\alpha^{(i-1)}$, the solution to $R_n^{i-1} \alpha^{i-1} = E_n^{i-1}$

we derive solution to $R_n^i \alpha^i = E_n^i$

The $(i - 1)^{st}$ solution can be expressed as:

$$\begin{bmatrix} R_n[0] & R_n[1] & R_n[2] & \cdots & R_n[i-1] \\ R_n[1] & R_n[0] & R_n[1] & \cdots & R_n[i-2] \\ R_n[2] & R_n[1] & R_n[0] & \cdots & R_n[i-3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_n[i-1] & R_n[i-2] & R_n[i-3] & \cdots & R_n[0] \end{bmatrix} \begin{bmatrix} 1 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ \vdots \\ -\alpha_{i-1}^{(i-1)} \end{bmatrix} = \begin{bmatrix} E_n^{i-1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Appending a 0 to vector α and multiplying by the matrix $R_n^{(i)}$ gives

$$\begin{bmatrix} R_n[0] & R_n[1] & R_n[2] & \cdots & R_n[i] \\ R_n[1] & R_n[0] & R_n[1] & \cdots & R_n[i-1] \\ R_n[2] & R_n[1] & R_n[0] & \cdots & R_n[i-2] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_n[i-1] & R_n[i-2] & R_n[i-3] & \cdots & R_n[1] \\ R_n[i] & R_n[i-1] & R_n[i-2] & \cdots & R_n[0] \end{bmatrix} \begin{bmatrix} 1 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ \vdots \\ -\alpha_{i-1}^{(i-1)} \\ 0 \end{bmatrix} = \begin{bmatrix} E_n^{i-1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ \gamma^{(i-1)} \end{bmatrix}$$

where

$$\gamma^{(i-1)} = R_n[i] - \sum_{j=1}^{i-1} \alpha_j^{(i-1)} R_n[i-j]$$

Toeplitz matrix has special symmetry we can reverse the order of the equations

$$\begin{bmatrix} R_n[0] & R_n[1] & R_n[2] & \cdots & R_n[i] \\ R_n[1] & R_n[0] & R_n[1] & \cdots & R_n[i-1] \\ R_n[2] & R_n[1] & R_n[0] & \cdots & R_n[i-2] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_n[i-1] & R_n[i-2] & R_n[i-3] & \cdots & R_n[1] \\ R_n[i] & R_n[i-1] & R_n[i-2] & \cdots & R_n[0] \end{bmatrix} \begin{bmatrix} 0 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ \vdots \\ -\alpha_{i-1}^{(i-1)} \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma^{(i-1)} \\ 0 \\ 0 \\ \vdots \\ 0 \\ E_n^{i-1} \end{bmatrix}$$

Combine the two sets of matrices with a multiplicative factor k_i

$$R_n^i \begin{bmatrix} 1 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ \vdots \\ -\alpha_{i-1}^{(i-1)} \\ 0 \end{bmatrix} - k_i \begin{bmatrix} 0 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ \vdots \\ -\alpha_{i-1}^{(i-1)} \\ 1 \end{bmatrix} = \begin{bmatrix} E_n^{i-1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ \gamma^{(i-1)} \end{bmatrix} - k_i \begin{bmatrix} \gamma^{(i-1)} \\ 0 \\ 0 \\ \vdots \\ 0 \\ E_n^{i-1} \end{bmatrix}$$

Choose of $\gamma^{(i-1)}$ so that vector on right has only a single non-zero entry, i.e

$$k_i = \frac{\gamma^{(i-1)}}{E^{i-1}} = \frac{R_n[i] - \sum_{j=1}^{i-1} \alpha_j^{(i-1)} R_n[i-j]}{E^{i-1}}$$

The first element of the right hand side vector is now

$$E_n^{(i)} = E_n^{(i-1)} - k_i \gamma^{(i-1)} = E_n^{(i-1)} (1 - k_i^2)$$

2

The k_i parameters are called Partial Correlation (PARCOR) coefficients

So the vector of order i^{th} predictor coefficients is

$$\begin{bmatrix} 1 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ \vdots \\ -\alpha_{i-1}^{(i-1)} \\ -\alpha_i^{(i)} \end{bmatrix} = \begin{bmatrix} 1 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ \vdots \\ -\alpha_{i-1}^{(i-1)} \\ 0 \end{bmatrix} - k_i \begin{bmatrix} 0 \\ -\alpha_1^{(i-1)} \\ -\alpha_2^{(i-1)} \\ \vdots \\ -\alpha_{i-1}^{(i-1)} \\ 1 \end{bmatrix}$$

$$\alpha_j^{(i)} = \alpha_j^{(i-1)} - k_i \alpha_{i-j}^{(i-1)} \quad 3$$

$$\alpha_i^{(i)} = k_i \quad 4$$

A. The final solution for order p is:

$$\alpha_j = \alpha_j^p \quad 1 \leq j \leq p$$

B. with prediction error

$$E_n^{(p)} = E_n[0] \prod_{m=1}^p (1 - k_m^2) = R_n[0] \prod_{m=1}^p (1 - k_m^2)$$

C. If we use normalized autocorrelation coefficients:

$$r_n[k] = \frac{R_n[k]}{R_n[0]}$$

D. normalized prediction error

$$v^{(i)} = \prod_{m=1}^i (1 - k_m^2) \quad 0 \leq v^{(i)} \leq 1 \quad -1 \leq k_i \leq 1$$

$$\mathcal{E}^{(0)} = R[0]$$

for $i = 1, 2, \dots, p$

$$k_i = \left(R[i] - \sum_{j=1}^{i-1} \alpha_j^{(i-1)} R[i-j] \right) / \mathcal{E}^{(i-1)}$$

$$\alpha_i^{(i)} = k_i$$

if $i > 1$ then for $j = 1, 2, \dots, i-1$

$$\alpha_j^{(i)} = \alpha_j^{(i-1)} - k_i \alpha_{i-j}^{(i-1)}$$

end

$$\mathcal{E}^{(i)} = (1 - k_i^2) \mathcal{E}^{(i-1)}$$

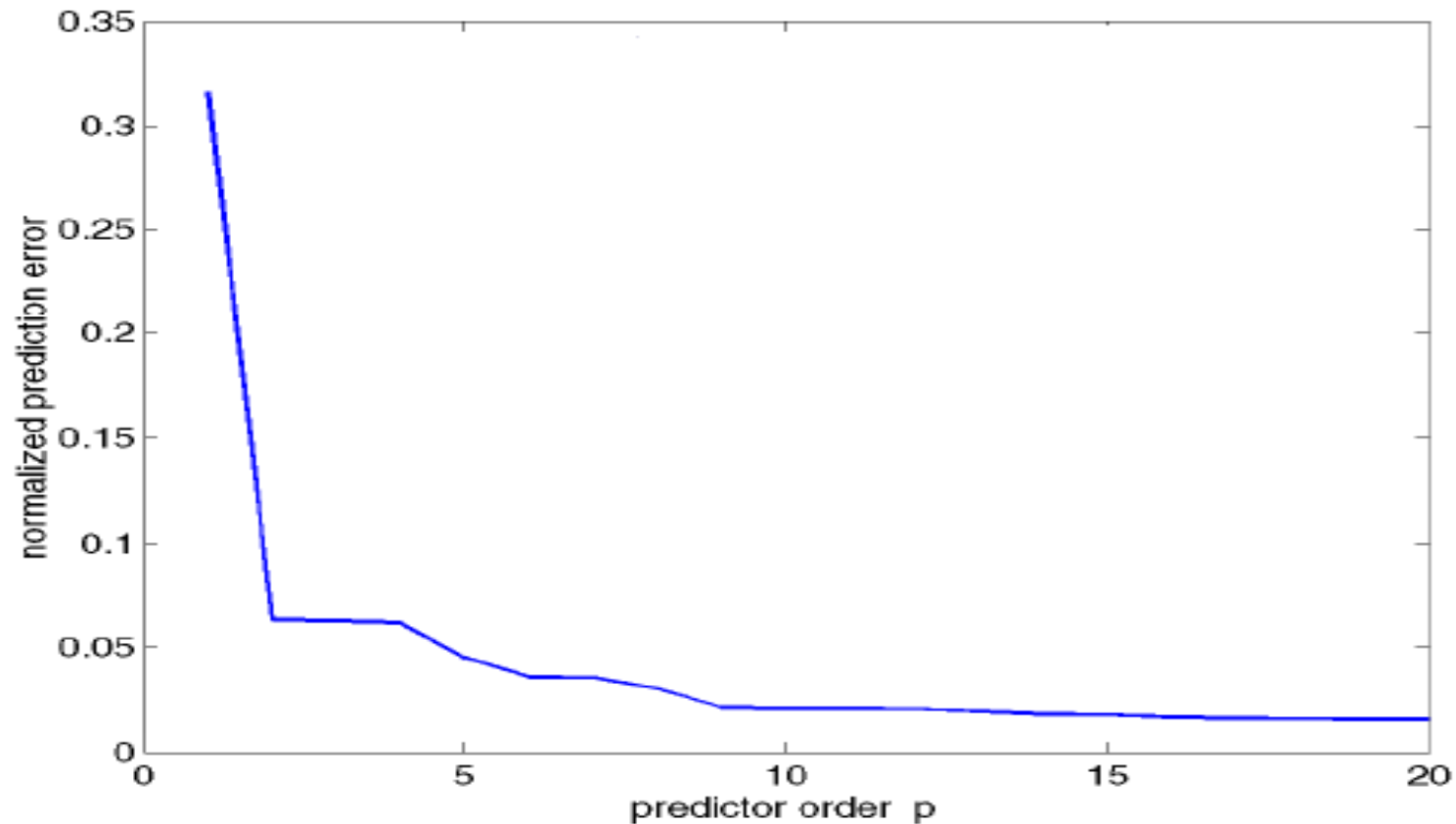
end

$$\alpha_j = \alpha_j^{(p)} \quad j = 1, 2, \dots, p$$



1. Consider a simple $p = 2$ *find the solution*
2. Consider a simple $p = 3$ *find the solution*
3. Consider a simple $p = 4$ *find the solution*

Prediction Error vs. Model Order



Model order is usually determined by:

- $F_s/1000$ poles for vocal tract
- 2-4 poles for radiation
- 2 poles for glottal pulse

Computation of Model Gain

$$H(z) = \frac{G}{1 - \sum_{k=1}^p \alpha_k z^{-k}}$$

$$h[m] = \sum_{k=1}^p \alpha_k h[m-k] + G \delta[m] \quad \text{1}$$

Multiply by $h[m-i]$

$$h[m]h[m-i] = \sum_{k=1}^p \alpha_k h[m-k]h[m-i] + G \delta[m]h[m-i]$$

let $i = 0$

$$\sum_{m=-\infty}^{\infty} h[m]h[m] = \sum_{k=1}^p \alpha_k \sum_{m=-\infty}^{\infty} h[m-k]h[m] + G \sum_{m=-\infty}^{\infty} \delta[m]h[m]$$

$$r_h[0] = \sum_{k=1}^p \alpha_k r_h[k] + G^2 \quad \text{since } h[0] = G \quad \text{Put } m=0 \text{ in equation (1)}$$



Energy in $h[m]$ = Energy in short time measurement $s_n[m]$

$$r_h[0] = r_n[0]$$

$$\text{and } r_h[\tau] = r_n[\tau] \quad \text{for } \tau \leq p$$

so

$$G^2 = r_h[0] - \sum_{k=1}^p \alpha_k r_h[k] = r_n[0] - \sum_{k=1}^p \alpha_k r_n[k] = E_n$$

Solution for Gain (Voiced)



Solution for Gain (Unvoiced)

For unvoiced speech the input is white noise with zero mean and unity variance,

Excite the system with input $Au[n]$

$$Y(z) = H(z)U(z)$$

$$H(z) = \frac{G}{1 - \sum_{k=1}^p \alpha_k z^{-k}}$$

$$Y(z) - \sum_{k=1}^p \alpha_k z^{-k} Y(z) = GU(z)$$

$$Y(z) = \sum_{k=1}^p \alpha_k z^{-k} Y(z) + GU(z)$$

$$y[m] = \sum_{k=1}^p \alpha_k y[m-k] + Gu[m]$$



Multiply by $y[m-i]$

$$y[m]y[m-i] = \sum_{k=1}^p \alpha_k y[m-k]y[m-i] + Gu[m]y[m-i]$$

let $i = 0$

$$\sum_{m=-\infty}^{\infty} y[m]y[m] = \sum_{k=1}^p \alpha_k \sum_{m=-\infty}^{\infty} y[m-k]y[m] + G \sum_{m=-\infty}^{\infty} u[m]y[m]$$

$$r_y[0] = \sum_{k=1}^p \alpha_k r_y[k] + G^2 \quad \text{since} \quad h[0] = G$$



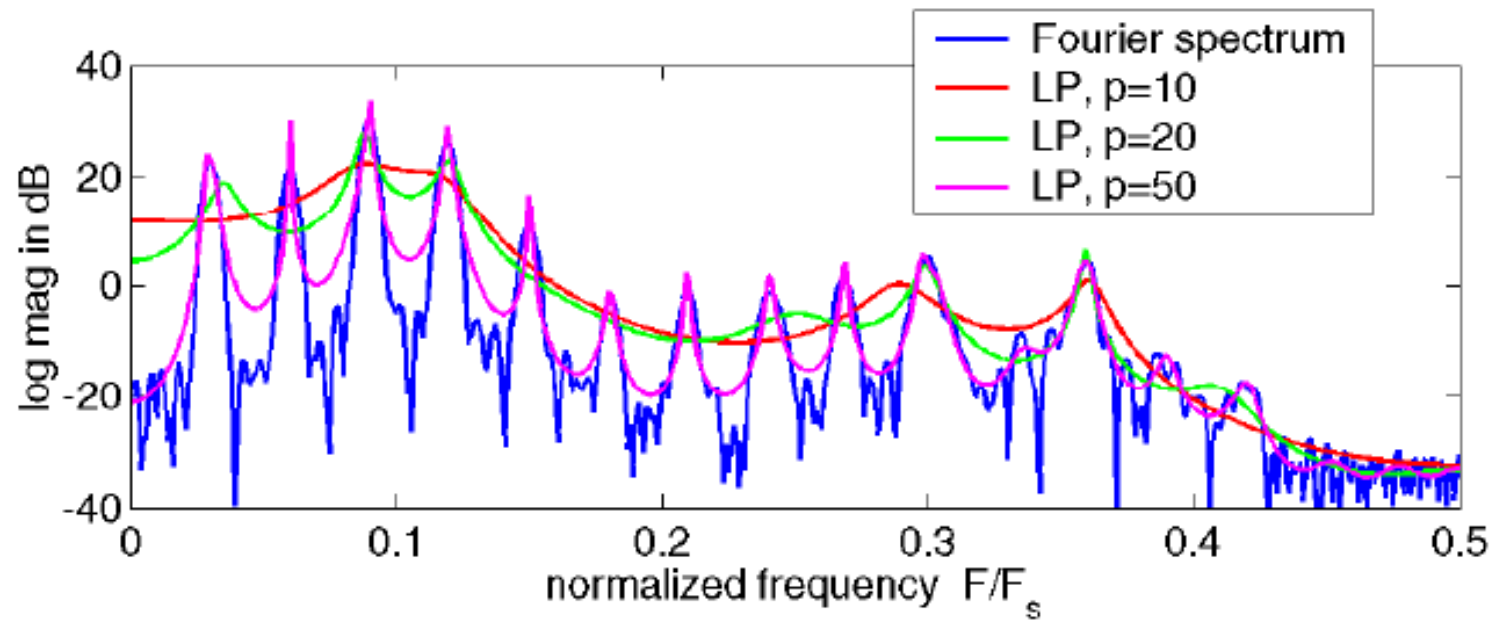
Frequency Domain Interpretations of Linear Predictive Analysis

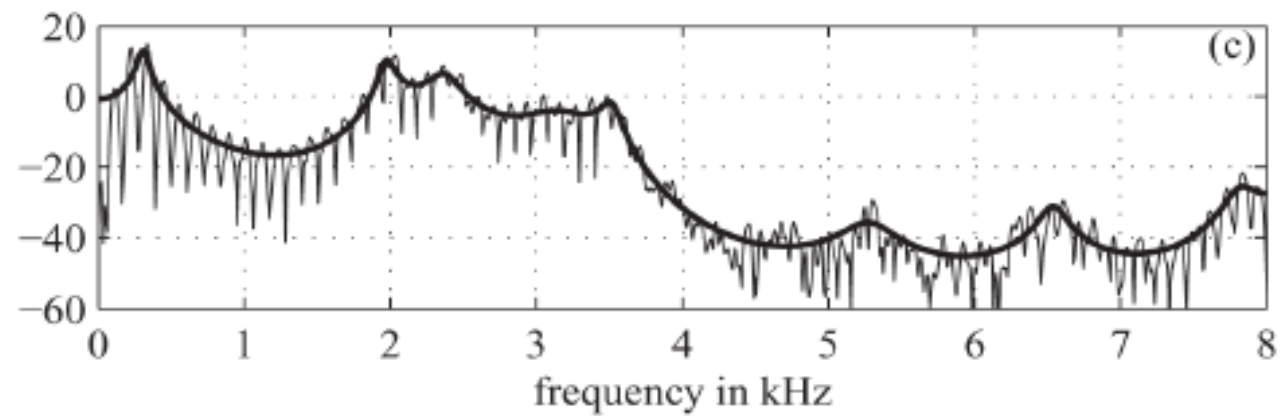
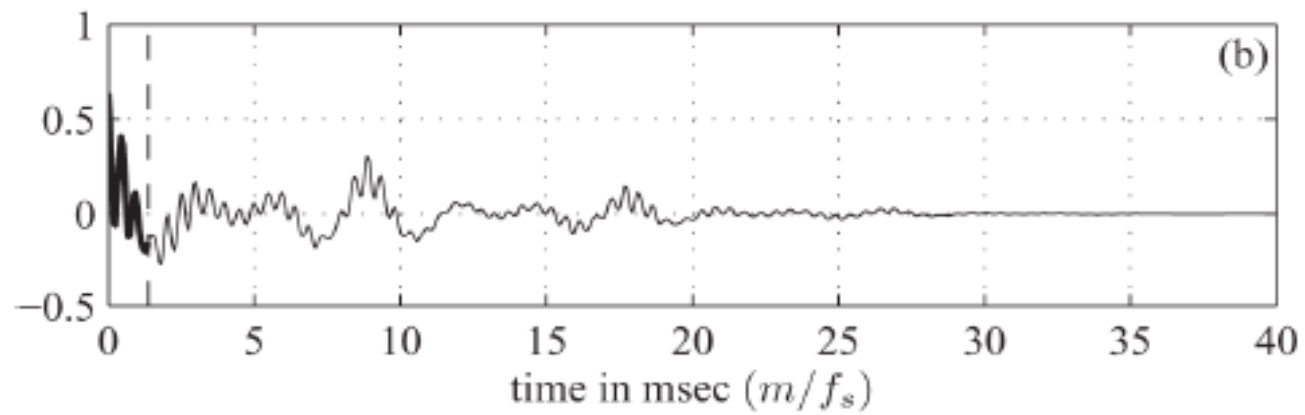
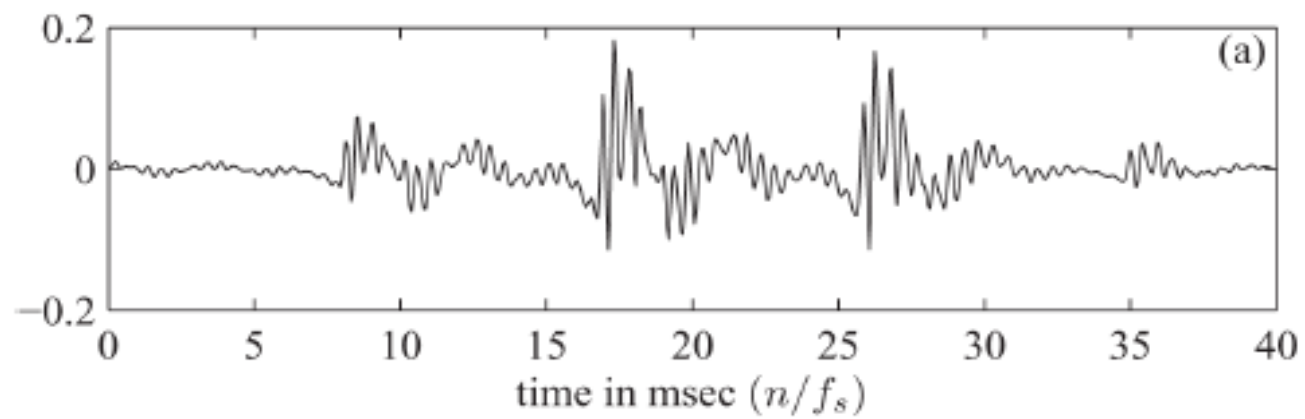
The final LPC model consists of the LPC parameters, α_k and gain A together which define the system function

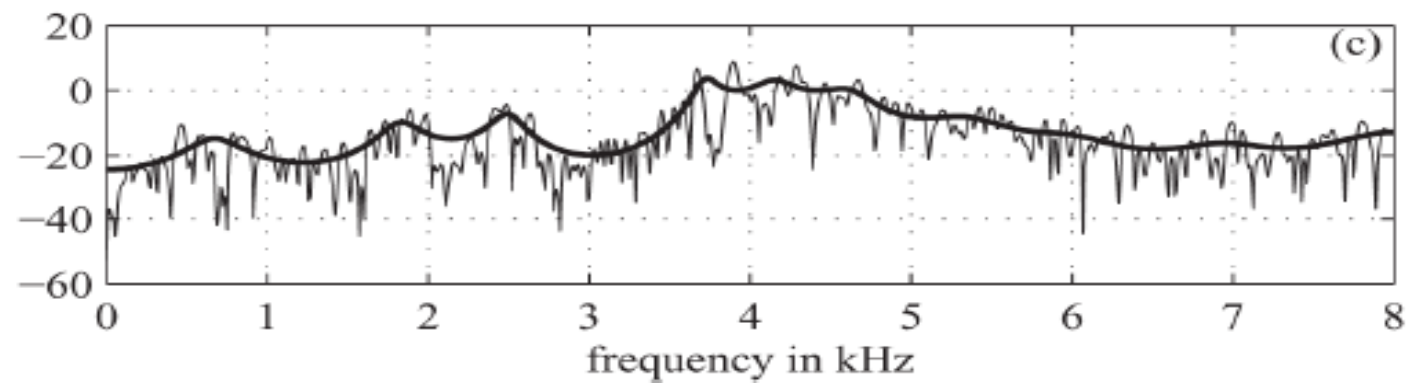
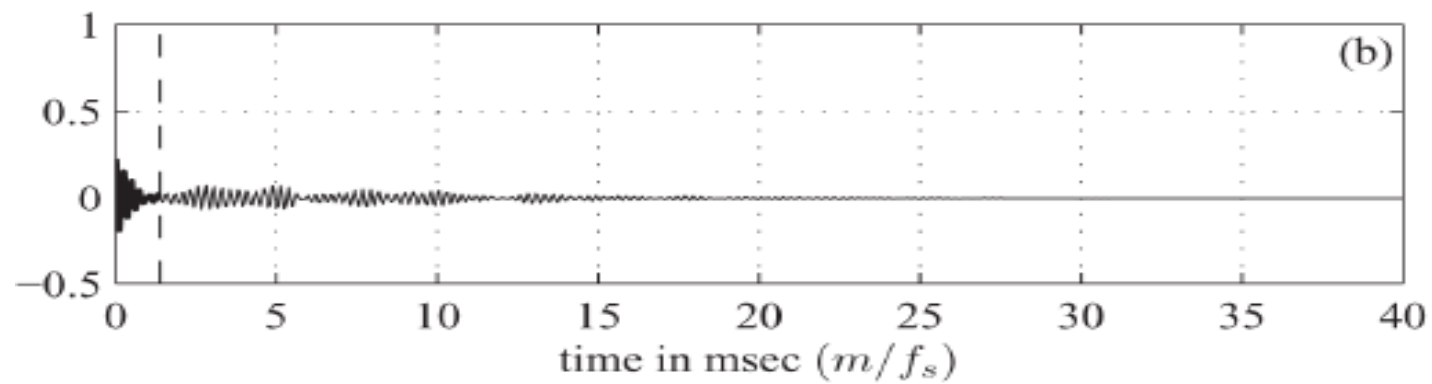
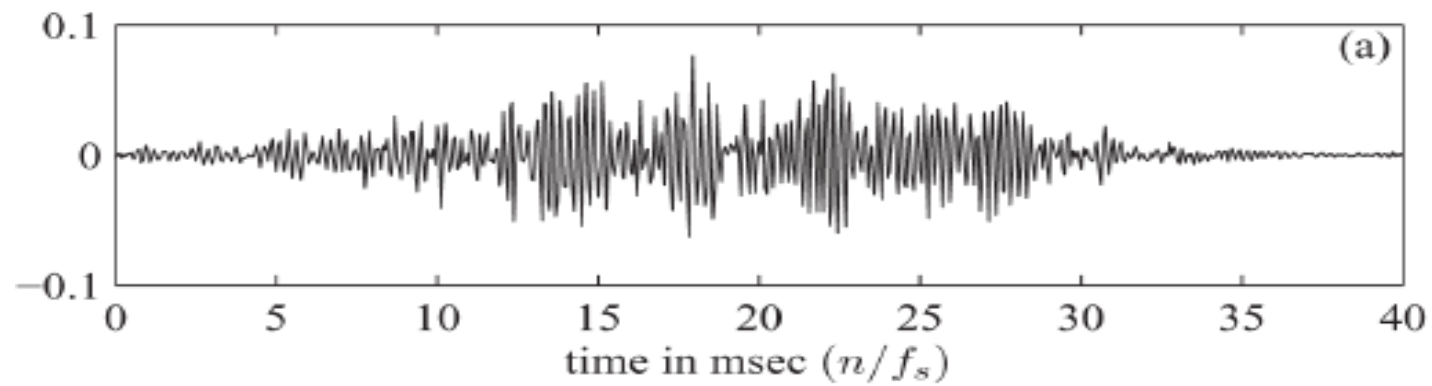
$$H(z) = \frac{G}{1 - \sum_{k=1}^p \alpha_k z^{-k}}$$

$$H(e^{j\omega}) = \frac{G}{1 - \sum_{k=1}^p \alpha_k e^{j\omega k}}$$

LPC Spectrum





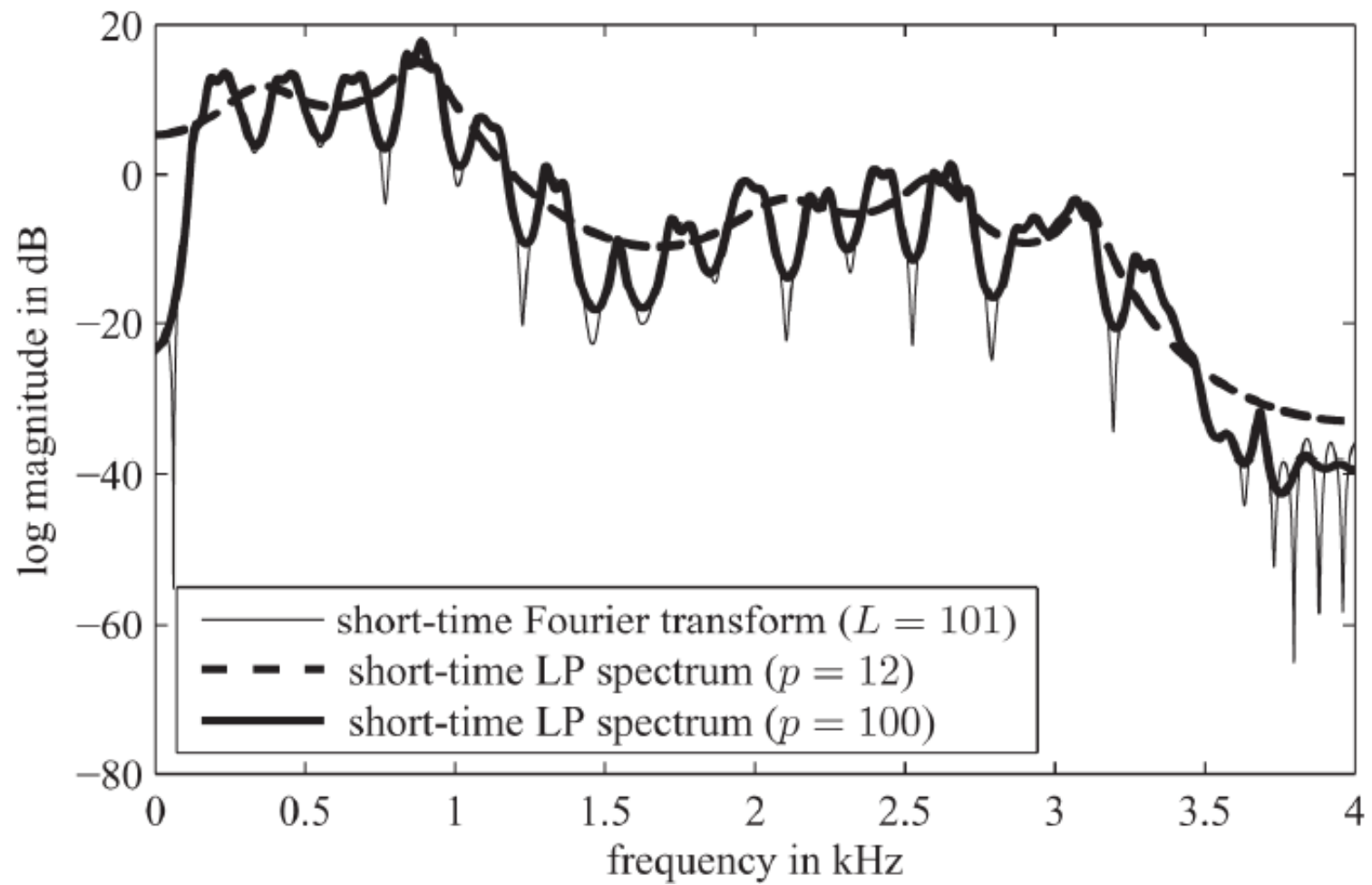


Effects of Model Order



The AC $R_n[m]$ function of the speech segment $S_n[m]$, and the AC function $\hat{R}_n[m]$, of the impulse response, corresponding to the system function $h_n[m]$, are equal for the first $(p + 1)$ values. As $p \rightarrow \infty$ the AC functions are equal for all values and thus:

$$\lim_{p \rightarrow \infty} \left| \hat{H}(e^{j\omega}) \right|^2 = \left| S_n(j\omega) \right|^2$$





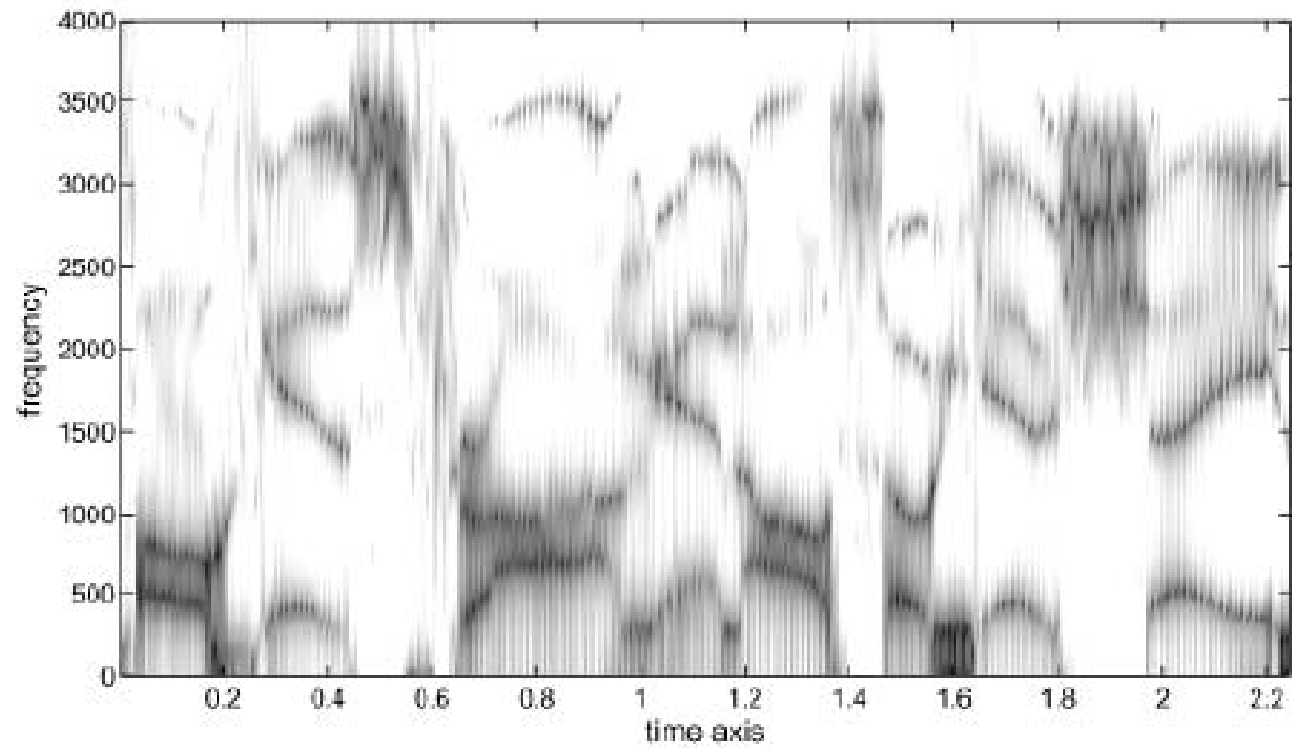
Linear Prediction Spectrogram

Speech spectrogram

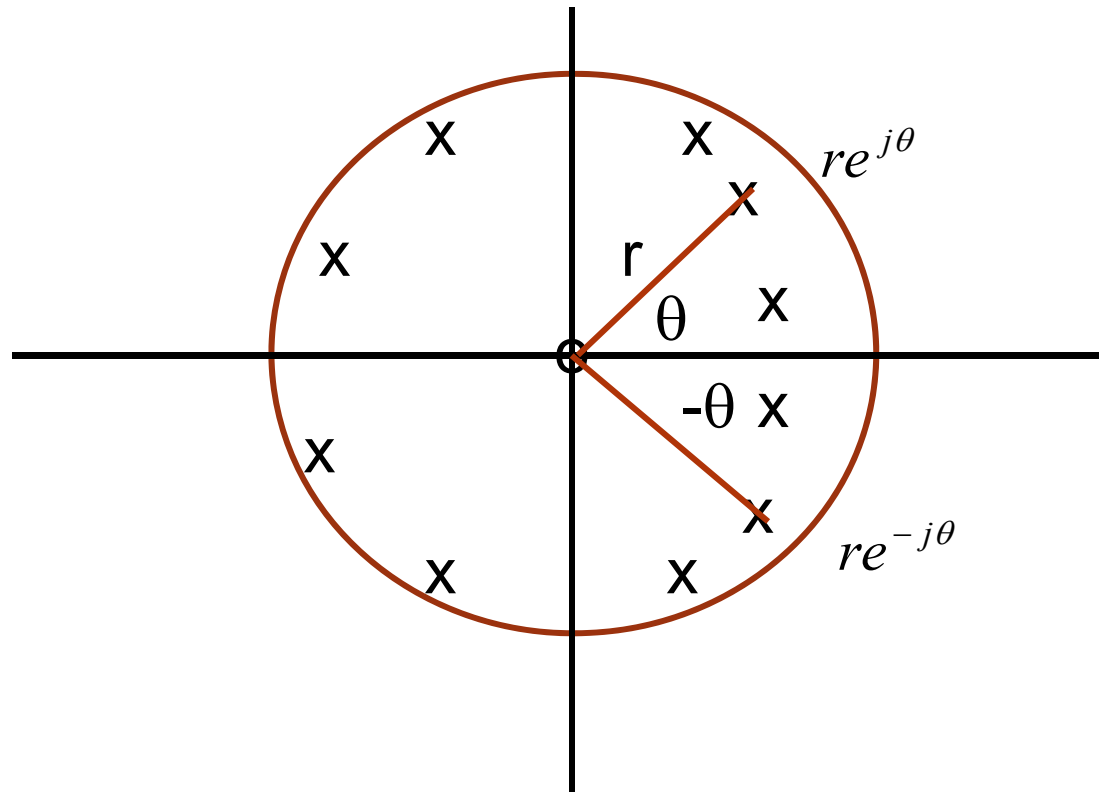
$$20 \log(|S_n(k)|) = 20 \log \left| \sum_{m=0}^{L-1} s_n[nR + m] w[m] e^{-j(2\pi mk / N)} \right|$$

LPC spectrogram

$$20 \log(|H_n(k)|) = 20 \log \frac{G}{\left| \sum_{m=0}^{L-1} A_n e^{j(2\pi mk / N)} \right|}$$



$$H(z) = \frac{G}{A(z)} = \frac{G}{1 - \sum_{k=1}^p \alpha_k z^{-k}} = \frac{Gz^p}{\prod_{k=1}^p (z - z_k)}$$





Angle corresponds to frequency, and radius corresponds to bandwidth. So we can determine the pole (or resonant) frequencies and bandwidths (converting to Hz)

$$\text{Frequency} = \frac{F_s}{2\pi} \theta$$

$$\text{Bandwidth} = -\log(r) \cdot \frac{F_s}{\pi} = -\log(\text{abs}(z_k)) \cdot \frac{F_s}{\pi}$$

Can find values of α_k that minimize by setting

$$\frac{\partial E_n}{\partial \alpha_i} = 0, \quad i = 1, 2, \dots, p$$

$$\begin{aligned} \frac{\partial E_n}{\partial \alpha_i} &= \frac{\partial}{\partial \alpha_i} \sum_{-\infty}^{\infty} (s_n[m] - \sum_{k=1}^p \alpha_k s_n[m-k])^2 \\ &= 2 \sum_{-\infty}^{\infty} (s_n[m] - \sum_{k=1}^p \alpha_k s_n[m-k]) \left(-\frac{\partial}{\partial \alpha_i} \sum_{k=1}^p \alpha_k s_n[m-k] \right) \end{aligned}$$

Where

$$-s_n[m-i] = -\frac{\partial}{\partial \alpha_i} \sum_{k=1}^p \alpha_k s_n[m-k]$$

$\alpha_k s_n[m-k]$ is constant with respect to $\frac{\partial}{\partial \alpha_i}$ for $k \neq i$

$$0 = 2 \sum_{-\infty}^{\infty} (s_n[m] - \sum_{k=1}^p \alpha_k s_n[m-k]) (-s_n[m-i])$$

$$\sum_{-\infty}^{\infty} s_n[m-i] s_n[m] = \sum_{k=1}^p \alpha_k \sum_{-\infty}^{\infty} s_n[m-i] s_n[m-k] \quad 1 \leq i \leq p \quad (1)$$

let

$$\phi_n[i, k] = \sum_m s_n[m-i] s_n[m-k] \quad 1 \leq i \leq p$$

then

$$\phi_n[i, 0] = \sum_{k=1}^p \alpha_k \phi_n[i, k] \quad i = 1, 2, \dots, p \quad (2)$$

leading to a set of p equations in p unknowns that can be solved in an efficient manner for the $\{\alpha_k\}$

Minimum mean-squared prediction error



$$E_n = \sum_{m=-\infty}^{\infty} e_n^2[m]$$

$$E_n = \sum_{m=-\infty}^{\infty} [s_n[m] - \sum_{k=1}^p \alpha_k s_n[m-k]]^2$$
$$= \sum_{m=-\infty}^{\infty} s_n^2[m] - 2 \sum_{m=-\infty}^{\infty} s_n[m] \sum_{k=1}^p \alpha_k s_n[m-k]$$

$$+ \sum_{m=-\infty}^{\infty} \sum_{k=1}^p \alpha_k s_n[m-k] \sum_{l=1}^p \alpha_l s_n[m-l]$$

$$= \sum_{m=-\infty}^{\infty} s_n^2[m] - \sum_{k=1}^p \alpha_k \sum_{m=-\infty}^{\infty} s_n[m-k] s_n[m]$$

$$= \varphi_n[0,0] - \sum_{k=1}^p \alpha_k \varphi_n[0,k]$$

$$\sum_{m=-\infty}^{\infty} \sum_{k=1}^p \alpha_k s_n[m-k] \sum_{l=1}^p \alpha_l s_n[m-l]$$
$$= \sum_{m=-\infty}^{\infty} \sum_{k=1}^p \alpha_k s_n[m-k] s_n[m]$$
$$= \sum_{m=-\infty}^{\infty} s_n[m] \sum_{k=1}^p \alpha_k s_n[m-k]$$



$$\phi_n[i, k] = \sum_m S_n[m - i] S_n[m - k] \quad 1 \leq i \leq p$$

$$\phi_n[i, k] = \sum_{m=0}^{L-1} S_n[m - i] S_n[m - k] \quad 1 \leq i \leq p \quad 0 \leq k \leq p$$

Changing the summation index gives

$$\phi_n[i, k] = \sum_{m=-i}^{L-i-1} S_n[m] S_n[m + i - k] \quad 1 \leq i \leq p \quad 0 \leq k \leq p$$

$$\phi_n[i, k] = \sum_{m=-k}^{L-k-1} S_n[m] S_n[m + k - i] \quad 1 \leq i \leq p \quad 0 \leq k \leq p$$

key difference from Autocorrelation Method is that limits of summation include terms before $m=0$. i.e window extends p samples backwards

since we are extending window backwards, don't need to taper it using window function since there is ***no transition at window edges***

$$\phi_n[i,0] = \sum_{k=1}^p \alpha_k \phi_n[i,k] \quad i = 1, 2, \dots, p \quad 1$$

$$E_n = \phi_n[0,0] - \sum_{k=1}^p \alpha_k \phi_n[0,k] \quad 2$$

$$\begin{bmatrix} \phi_n[1,1] & \phi_n[1,2] & \phi_n[1,3] & \cdots & \phi_n[1,p] \\ \phi_n[2,1] & \phi_n[2,2] & \phi_n[2,3] & \cdots & \phi_n[2,p] \\ \phi_n[3,1] & \phi_n[3,2] & \phi_n[3,3] & \cdots & \phi_n[3,p] \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_n[p,1] & \phi_n[p,2] & \phi_n[p,3] & \cdots & \phi_n[p,p] \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} \phi_n[1,0] \\ \phi_n[2,0] \\ \phi_n[3,0] \\ \vdots \\ \phi_n[p,0] \end{bmatrix}$$

$$\phi \alpha = \psi$$

$$\alpha = \phi^{-1} \psi$$



the solution of the matrix equation is called the Cholesky decomposition, or square root method

ϕ Matrix is symmetric but not diagonal elements are same

$$\phi = ADA^t$$

Where A = lower triangular matrix with 1's on the main diagonal;
D=diagonal matrix

determine elements of A and D by solving for (i, j) elements of the matrix equation

Let $p=4$ and matrix element $\phi_n(i, j) = \phi_{ij}$

$$\begin{bmatrix} \phi_{11} & \phi_{21} & \phi_{31} & \phi_{41} \\ \phi_{21} & \phi_{22} & \phi_{32} & \phi_{42} \\ \phi_{31} & \phi_{32} & \phi_{33} & \phi_{43} \\ \phi_{41} & \phi_{42} & \phi_{43} & \phi_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ A_{21} & 1 & 0 & 0 \\ A_{31} & A_{32} & 1 & 0 \\ A_{41} & A_{42} & A_{43} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix} \begin{bmatrix} 1 & A_{21} & A_{31} & A_{41} \\ 0 & 1 & A_{32} & A_{42} \\ 0 & 0 & 1 & A_{43} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve matrix

$$d_j = \varphi_{jj} - \sum_{k=1}^{j-1} A_{jk}^2 d_k$$

$$A_{i1} = \phi_{i1} / d_1$$

Else

$$A_{ij} = \varphi_{ij} - \sum_{k=1}^{j-1} A_{ik} d_k A_{jk} / d_j$$

$$d_p = \varphi_{pp} - \sum_{k=1}^{p-1} A_{pk}^2 d_k$$

$$d_1 = \varphi_{11}$$

$$A_{21} d_1 = \varphi_{21} \rightarrow A_{21} = \frac{\varphi_{21}}{d_1}$$

$$A_{31} d_1 = \varphi_{31} \rightarrow A_{31} = \frac{\varphi_{31}}{d_1}$$

$$A_{41} d_1 = \varphi_{41} \rightarrow A_{41} = \frac{\varphi_{41}}{d_1}$$

$$d_2 = \varphi_{22} - A_{21}^2 d_1$$

$$A_{32} d_2 = \varphi_{32} - A_{31} d_1 A_{21} \rightarrow A_{32} = \frac{\varphi_{32} - A_{31} d_1 A_{21}}{d_2}$$

$$A_{42} d_2 = \varphi_{42} - A_{41} d_1 A_{21} \rightarrow A_{42} = \frac{\varphi_{42} - A_{41} d_1 A_{21}}{d_2}$$

iterate procedure to solve for d_3, A_{43}, d_4

```
d1=phi11
for(i=1; i<=p; i++)
{
    A_i1=phi_i1/d1
}
```

```
for(j=2; j<p; j++)
{
    d_j = phi_jj - sum_{k=1}^{j-1} A_jk^2 d_k
    for(i=j+1; i<=p; i++)
    {
        A_ij = phi_ij - sum_{k=1}^{j-1} A_ik d_k A_jk / d_j
    }
}
```



$$ADA^t \alpha = \psi$$

$$\text{Let } AY = \psi$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ A_{21} & 1 & 0 & 0 \\ A_{31} & A_{32} & 1 & 0 \\ A_{41} & A_{42} & A_{43} & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}$$

$$Y_1 = \psi_1$$

$$Y_2 = \psi_2 - A_{21} Y_1$$

$$Y_3 = \psi_3 - A_{31} Y_1 - A_{32} Y_2$$

$$Y_4 = \psi_4 - A_{41} Y_1 - A_{42} Y_2 - A_{43} Y_3$$

$$Y_i = \psi_i - \sum_{j=1}^{i-1} A_{ij} Y_j$$

```
Y1=ψ1
for(i=1; i<=p; i++)
{
    Yi = ψi - ∑j=1i-1 Aij Yj
}
```

$$\Rightarrow DA^t \alpha = Y \Rightarrow A^t \alpha = D^{-1} Y$$

$$\begin{bmatrix} 1 & A_{21} & A_{31} & A_{41} \\ 0 & 1 & A_{32} & A_{42} \\ 0 & 0 & 1 & A_{43} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 1/d_1 & 0 & 0 & 0 \\ 0 & 1/d_2 & 0 & 0 \\ 0 & 0 & 1/d_3 & 0 \\ 0 & 0 & 0 & 1/d_4 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}$$

$$\alpha_4 = \frac{Y_4}{d_4}$$

$$\alpha_3 = \frac{Y_3}{d_3} - A_{43} \alpha_4$$

$$\alpha_2 = \frac{Y_2}{d_2} - A_{43} \alpha_4 - A_{32} \alpha_3$$

$$\alpha_1 = \frac{Y_1}{d_1} - A_{43} \alpha_4 - A_{32} \alpha_3 - A_{21} \alpha_2$$

$$\alpha_i = \frac{y_i}{d_i} - \sum_{j=i+1}^p A_{ji} \alpha_j$$

$\alpha_p = Y_p / d_p$
for(i=1; i<=p; i++)

{
$$\alpha_i = \frac{y_i}{d_i} - \sum_{j=i+1}^p A_{ji} \alpha_j$$

}

Calculation proceeds backwards from $i=p-1$ to 1



Both covariance and autocorrelation methods of LP use two step solutions

Step-1: computation of a matrix of correlation values

Step-2: efficient solution of a set of linear equations

In lattice methods, two steps are combined into a recursive algorithm for determining LP parameters

It begin with Durbin algorithm--at the i^{th} stage the set of coefficients are coefficients of the i^{th} order optimum LP



$$H(z) = \frac{A}{1 - \sum_{k=1}^p \alpha_k z^{-k}}$$

$$A(z) = 1 - \sum_{k=1}^p \alpha_k z^{-k}$$

i^{th} order prediction error filter

$$A^i(z) = 1 - \sum_{k=1}^i \alpha_k z^{-k}$$

$$\underbrace{e^i[m]}_{\text{Forward prediction error}} = s[m] - \tilde{s}[m] = s[m] - \sum_{k=1}^i \alpha_k^i s[m-k]$$

Forward prediction error

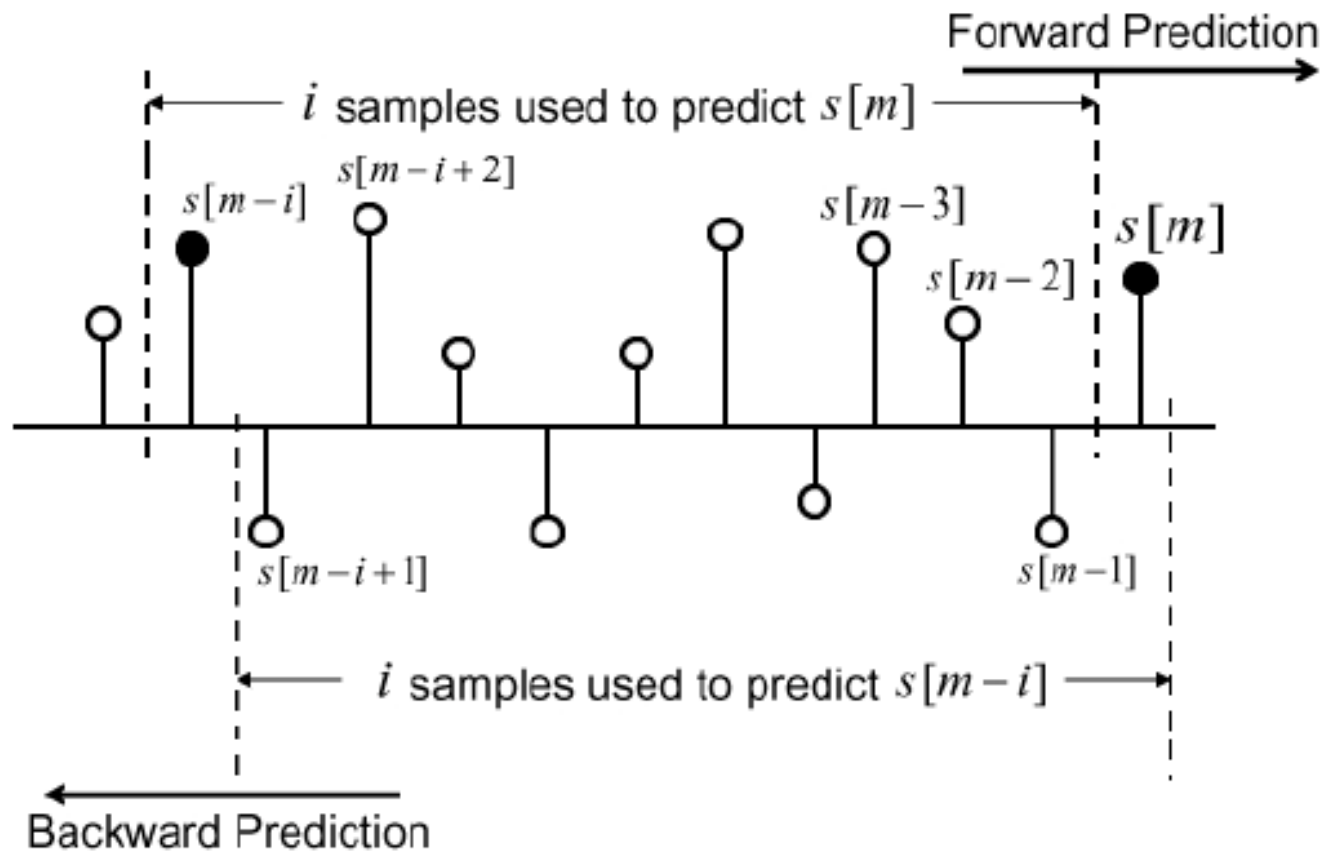
$$E^i(z) = A^i(z)S(z)$$

$$b^i[m] = s[m-i] - \sum_{k=1}^i \alpha_k^i s[m-i+k]$$

$$B^i(z) = z^{-i}S(z) - \sum_{k=1}^i \alpha_k^i z^{-i} z^k S(z) = z^{-i}S(z) \left[1 - \sum_{k=1}^i \alpha_k^i z^k \right]$$

$$B(z) = z^{-i}S(z)A^i(z^{-1})$$

Lattice Formulations of LP



same set of samples is used to forward predict $s(m)$ and backward predict $s(m-i)$



Levinson Recursion

Step-1

$$E^{(0)} = r(0)$$

$$\alpha_0^0 = 0$$

Step-2

Weighting factor of i^{th} pole model

$$k_i = \left\{ r(i) - \sum_{j=1}^{i-1} \alpha_j^{(i-1)} r(i-j) \right\} / E^{(i-1)}, \quad 1 \leq i \leq p$$

Step-3

$$\alpha_i^{(i)} = k_i$$

$$\alpha_j^{(i)} = \alpha_j^{(i-1)} - k_i \alpha_{i-j}^{(i-1)}$$

Step-4

Update the mean square prediction error

$$E^{(i)} = (1 - k_i^2) E^{(i-1)}$$

$$\alpha_i^i = k_i$$

$$\alpha_j^i = \alpha_j^{i-1} - k_i \alpha_{i-j}^{i-1}$$

$$A^i(z) = 1 - \sum_{k=1}^i \alpha_k z^{-k}$$

$$\begin{aligned} A^i(z) &= 1 - \sum_{k=1}^{i-1} \alpha_k^{i-1} z^{-k} - \alpha_i^i z^{-i} = 1 - \sum_{k=1}^{i-1} \alpha_k^{i-1} z^{-k} - k_i z^{-i} \\ &= 1 - \sum_{k=1}^{i-1} [\alpha_k^{i-1} z^{-k} - k_i \alpha_{i-k}^{i-1} z^{-k}] - k_i z^{-i} \\ &= [1 - \sum_{k=1}^{i-1} \alpha_k^{i-1} z^{-k}] + k_i \sum_{k=1}^{i-1} \alpha_{i-k}^{i-1} z^{-k} - k_i z^{-i} \\ &= [1 - \sum_{k=1}^{i-1} \alpha_k^{i-1} z^{-k}] + k_i \sum_{k'=i-1}^i \alpha_k^{i-1} z^{-i+k} - k_i z^{-i} \quad \text{put } k' = i - k \\ &= [1 - \sum_{k=1}^{i-1} \alpha_k^{i-1} z^{-k}] - k_i z^{-i} [1 - \sum_{k'=i-1}^i \alpha_k^{i-1} z^{k'}] \end{aligned}$$

$$A^{i-1}(z)$$

$$A^{i-1}(z^{-1})$$



$$A^i(z) = A^{i-1}(z) - k_i z^{-i} A^{i-1}(z^{-1})$$

$$\begin{aligned} E^i(z) &= A^i(z)S(z) = A^{i-1}(z)S(z) - k_i z^{-i} A^{i-1}(z^{-1})S(z) \\ &= E^{i-1}(z) - k_i z^{-1} B^{i-1}(z) \end{aligned}$$

$$e^i[m] = e^{i-1}[m] - k_i b^{i-1}[m-1]$$

$$B^i(z) = z^{-i} S(z) A^i(z^{-1}) = z^{-1} B^{i-1}(z) - k_i E^{i-1}(z)$$

$$b^i[m] = b^{i-1}[m-1] - k_i e^{i-1}[m]$$

$$e^i[m] = e^{i-1}[m] - k_i b^{i-1}[m-1]$$

Equation No -(1)

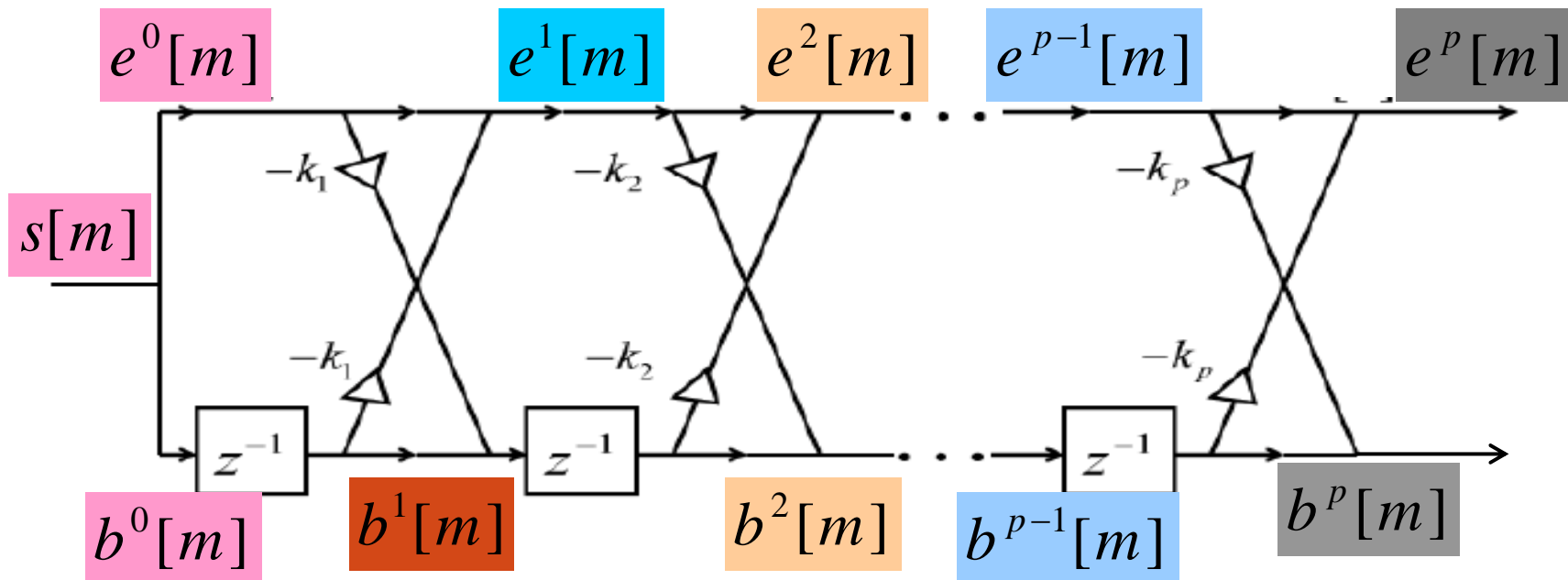
$$b^i[m] = b^{i-1}[m-1] - k_i e^{i-1}[m]$$

Equation No -(2)

$$e^0[m] = b^0[m] = s[m]$$

Equation No -(3)

No prediction

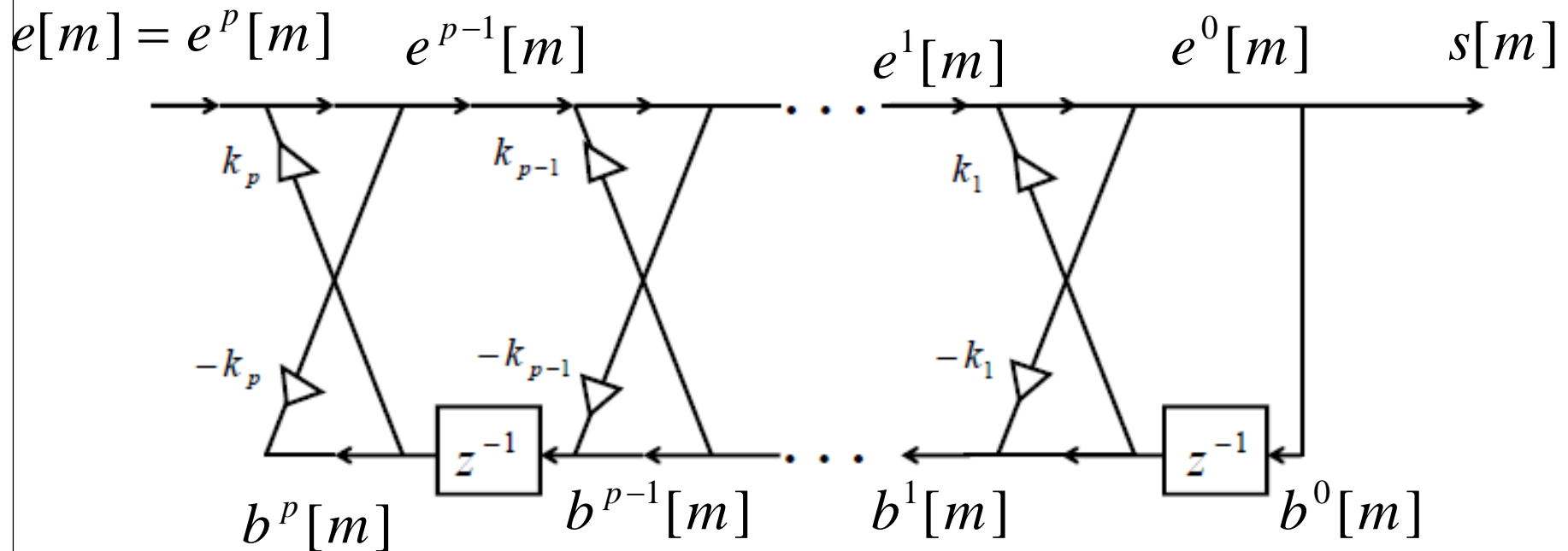


Lattice Filter for

$$A(Z)$$

$$e[m] = e^p[m]$$

All-Pole Lattice Filter for $H(z)$



$$e^{i-1}[m] = e^i[m] + k_i b^{i-1}[m-1]$$

$$b^{i-1}[m-1] = b^i[m] + k_i e^{i-1}[m]$$

$$H(z) = \frac{1}{A(z)}$$

$$S(z) = \frac{1}{A(z)} E(z)$$

All-Pole Lattice Filter for $H(z)$

1. since $b^{(i)}[-1] = 0, \forall i$, we can first solve for $e^{(i-1)}[0]$ for $i = p, p-1, \dots, 1$, using the relationship: $e^{(i-1)}[0] = e^{(i)}[0]$

2. since $b^{(0)}[0] = e^{(0)}[0]$ we can then solve for $b^{(i)}[0]$ for $i = 1, 2, \dots, p$ using the equation: $b^{(i)}[0] = -k_i e^{(i-1)}[0]$

3. we can now begin to solve for $e^{(i-1)}[1]$ as:

$$e^{(i-1)}[1] = e^{(i)}[1] + k_i b^{(i-1)}[0], i = p, p-1, \dots, 1$$

4. we set $b^{(0)}[1] = e^{(0)}[1]$ and we can then solve for $b^{(i)}[1]$ for $i = 1, 2, \dots, p$ using the equation:

$$b^{(i)}[1] = -k_i e^{(i-1)}[1] + b^{(i-1)}[0], i = 1, 2, \dots, p$$

5. we iterate for $n = 2, 3, \dots, N-1$ and end up with

$$s[n] = e^{(0)}[n] = b^{(0)}[n]$$

Direct Computation of k Parameters

$$e^i[m] = e^{i-1}[m] - k_i b^{i-1}[m-1]$$

Minimize forward prediction error as

$$E_{forward}^i = \sum_{m=0}^{L-1+i} [e^i[m]]^2 = \sum_{m=0}^{L-1+i} [e^{i-1}[m] - k_i b^{i-1}[m-1]]^2$$

$$\frac{\partial E_{forward}^i}{\partial k_i} = 0 = -2 \sum_{m=0}^{L-1+i} [e^{i-1}[m] - k_i b^{i-1}[m-1]] b^{i-1}[m-1]$$

$$k_i^{forward} = \frac{\sum_{m=0}^{L-1+i} e^{i-1}[m] b^{i-1}[m-1]}{\sum_{m=0}^{L-1+i} [b^{i-1}[m-1]]^2}$$

$$b^i[m] = b^{i-1}[m-1] - k_i e^{i-1}[m]$$

Minimize backward prediction error as

$$E_{backward}^i = \sum_{m=0}^{L-1+i} [b^i[m]]^2 = \sum_{m=0}^{L-1+i} [b^{i-1}[m-1] - k_i e^{i-1}[m]]^2$$

$$\frac{\partial E_{backward}^i}{\partial k_i} = 0 = -2 \sum_{m=0}^{L-1+i} [b^{i-1}[m-1] - k_i e^{i-1}[m]] e^{i-1}[m]$$

$$k_i^{backward} = \frac{\sum_{m=0}^{L-1+i} e^{i-1}[m] b^{i-1}[m-1]}{\sum_{m=0}^{L-1+i} [e^{i-1}[m]]^2}$$

$$\sum_{m=0}^{L-1+i} [e^{i-1}[m]]^2 = \sum_{m=0}^{L-1+i} [b^{i-1}[m-1]]^2$$

$$k_i^{\text{PARCOR}} = \sqrt{k_i^{\text{forward}} k_i^{\text{backward}}} = k_i^{\text{forward}} = k_i^{\text{backward}}$$

$$= \frac{\sum_{m=0}^{L-1+i} e^{i-1}[m] b^{i-1}[m-1]}{\left(\sum_{m=0}^{L-1+i} [e^{i-1}[m]]^2 \sum_{m=0}^{L-1+i} [b^{i-1}[m-1]]^2 \right)^{1/2}}$$

Direct Computation of k Parameters

- minimize **sum** of forward and backward prediction errors over fixed interval (covariance method)

$$E_{Burg}^{(i)} = \sum_{m=0}^{L-1} \left\{ \left[e^{(i)}(m) \right]^2 + \left[b^{(i)}(m) \right]^2 \right\}$$

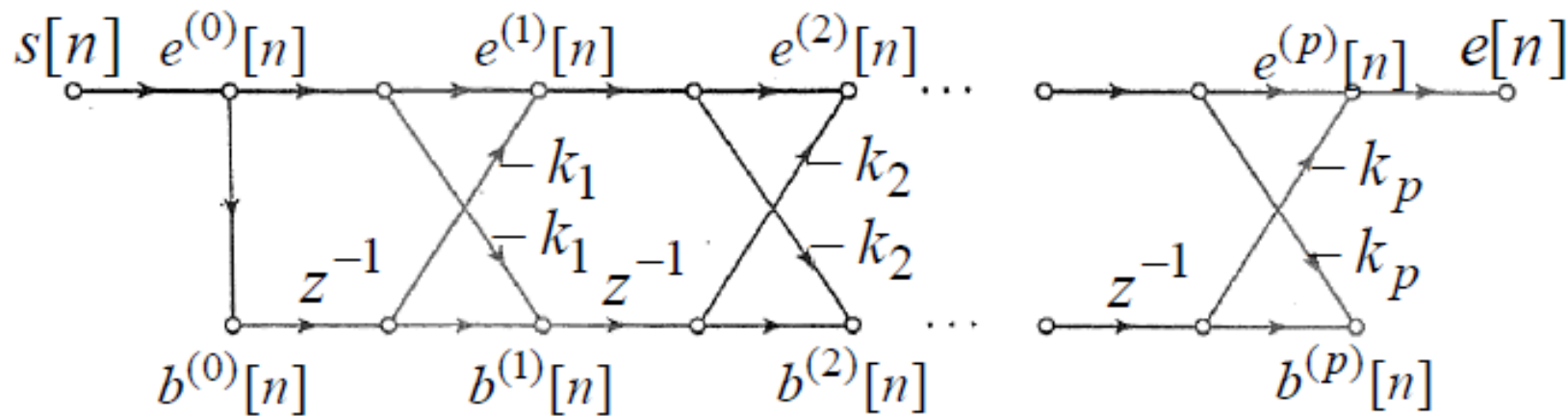
$$= \sum_{m=0}^{L-1} \left[e^{(i-1)}(m) - k_i b^{(i-1)}(m-1) \right]^2 + \sum_{m=-\infty}^{\infty} \left[-k_i e^{(i-1)}(m) + b^{(i-1)}(m-1) \right]^2$$

$$\frac{\partial E_{Burg}^{(i)}}{\partial k_i} = 0 = -2 \sum_{m=0}^{L-1} \left[e^{(i-1)}(m) - k_i b^{(i-1)}(m-1) \right] b^{(i-1)}(m-1)$$

$$- 2 \sum_{m=0}^{L-1} \left[-k_i e^{(i-1)}(m) + b^{(i-1)}(m-1) \right] e^{(i-1)}(m)$$

$$k_i^{Burg} = \frac{2 \sum_{m=0}^{L-1} \left[e^{(i-1)}(m) \cdot b^{(i-1)}(m-1) \right]}{\sum_{m=0}^{L-1} \left[e^{(i-1)}(m) \right]^2 + \sum_{m=0}^{L-1} \left[b^{(i-1)}(m-1) \right]^2}$$

- $-1 \leq k_i^{Burg} \leq 1$ **always**



$$e^{(0)}(m) = b^{(0)}(m) = s(m)$$

*3

$$k_i = \frac{\sum_{m=0}^{L-1+i} e^{(i-1)}(m)b^{(i-1)}(m-1)}{\left\{ \left[\sum_{m=0}^{L-1+i} [e^{(i-1)}(m)]^2 \right] \left[\sum_{m=0}^{L-1+i} [b^{(i-1)}(m-1)]^2 \right] \right\}^{1/2}}$$

*4

$$e^{(i)}(m) = e^{(i-1)}(m) - k_i b^{(i-1)}(m-1)$$

*1

$$b^{(i)}(m) = b^{(i-1)}(m-1) - k_i e^{(i-1)}(m)$$

*2

Lattice Algorithms

$$\mathcal{E}^{(0)} = R[0] \quad (1)$$

$$e^{(0)}[n] = b^{(0)}[n] = s[n], \quad 0 \leq n \leq L - 1 \quad (2)$$

for $i = 1, 2, \dots, p$

$$\text{compute } k_i \text{ using either Eq. (9.125) or Eq. (9.128)} \quad (3)$$

$$\text{compute } e^{(i)}[n], \quad 0 \leq n \leq L - 1 + i \text{ using Eq. (9.117b)} \quad (4a)$$

$$\text{compute } b^{(i)}[n], \quad 0 \leq n \leq L - 1 + i \text{ using Eq. (9.117c)} \quad (4b)$$

$$\alpha_i^{(i)} = k_i \quad (5)$$

compute predictor coefficients

if $i > 1$ then for $j = 1, 2, \dots, i - 1$

$$\alpha_j^{(i)} = \alpha_j^{(i-1)} - k_i \alpha_{i-j}^{(i-1)} \quad (6)$$

end

compute mean-squared energy

$$\mathcal{E}^{(i)} = (1 - k_i^2) \mathcal{E}^{(i-1)} \quad (7)$$

end

$$\alpha_j = \alpha_j^{(p)} \quad j = 1, 2, \dots, p \quad (8)$$

$$e[n] = e^{(p)}[n], \quad 0 \leq n \leq L - 1 + p \quad (9)$$