Dice: $S = \{1, 2, 3, 4, 5, 6\}$ $\qquad p(3) = 1/6$ "Unbiased die"

Coin: $S = \{H, T\}$ $\qquad p(H) = 1/2$ "Unbiased coin"

Axioms for Probability:-

(i) $p(E) > 0$ "Positivity"

(ii) $p(E_1) + p(E_2) \equiv$ "Probability of having either E_1 or E_2"
(E_1 and E_2 are disconnected)

$E_1 = H \quad , \quad E_2 = T$

$p(H) + p(T) = 1$

(iii) $\sum_{i=1}^{N} p(E_i) = 1$ "Normalization"

$p(H) + p(T) = 1$ "Coin"

$p(1) + p(2) + \cdots p(6) = 1$ "Die"

Q. How does one measure probability of any event?

Objective

$$p(E=4) = \lim_{N \to \infty} \frac{N_4}{N}$$

N_4 = # times you get '4'

N = total times you tossed the die.

Subjective.

Since face is unbiased! No experiment is needed

$$p(E=4) = \frac{1}{6}$$

$$\lim_{N \to \infty} \frac{N_4}{N} \longrightarrow \frac{1}{6}$$

Continuous random variables:

$S_x = \{a, b\}$ $\quad a < x < b$ "Bounded interval"

$S_x \equiv$ Outcome of measurement on 'x'

$S_x = \{-\infty, \infty\}$ "Unbounded interval"

Discrete probabilities: $\sum_{i=1}^{N} p(x_i) = 1$. $p(x)$ are dimensionless

Continuous probabilities: $\int_{-a}^{+a} p(x)\, dx = 1$ $\quad S_x \in \{-a, a\}$

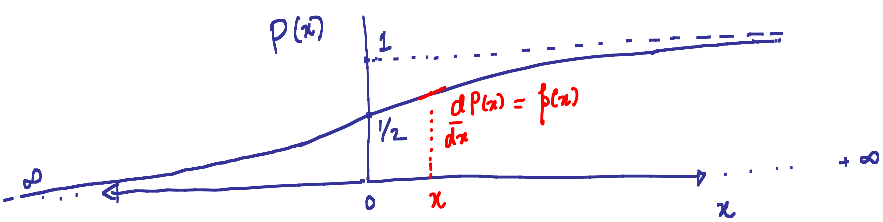
$p(x)$ are now dimensional $(1/x)$

$p(x)$ are now called as Probability density

Q. How do you construct Probability density functions (PDF) :

A. Needs a notion of Cumulative Probability function (CPF) : $P(x)$

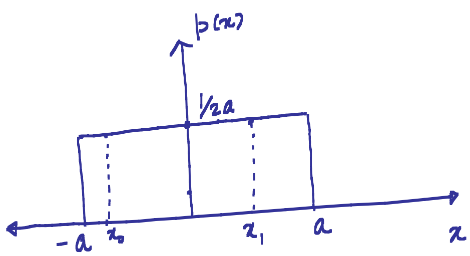
$P(x)$ is prob. of getting a $x \in [-\infty, x]$ if $x \in [-\infty, +\infty]$



Then you can construct $\boxed{p(x) = \frac{d P(x)}{dx}}$

Examples:

Flat distribution



$$\int_{-a}^{a} p(x)\,dx = 1$$

$$p(x) = \frac{1}{2a} \quad \because \; p(x) \text{ is const.}$$

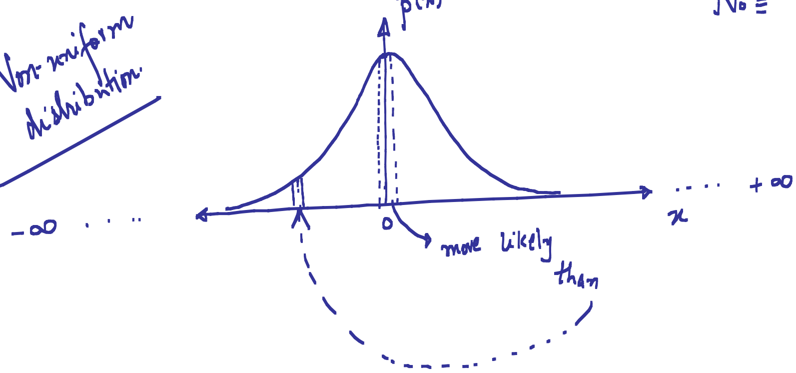
All measurements $x \in [-a, +a]$ are equally likely.

Non-uniform distribution

$$p(x) = N_0\, e^{-ax^2}$$

$N_0 \equiv$ Factor that normalizes the Gaussian PDF

$$\int_{-\infty}^{+\infty} N_0\, e^{-ax^2}\, dx = 1$$

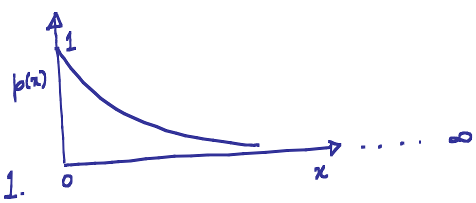


measurement on x is more likely to be near origin

* Mapping one distribution to another

Continuous random variable 'x' : $p(x)$ is the PDF

Say $p(x) = e^{-x}$



$$\text{Norm} = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1.$$

Consider a new random variable $y = x^2$

Q. What is the PDF of y? $p_y(y)$

A: Relies on conservation of norm.

$$p(x)\,dx \equiv \text{Probability of getting } x \in [x, x+dx]$$
$$= p_y(y)\,dy$$

$\downarrow$

PDF of y

= Probability of getting $y \in [y, y+dy]$

$$\underbrace{p(x)}_{\checkmark}\, dx = \underbrace{p_y(y)}_{?}\, dy \quad \text{——— (1)}$$

Knowing $\underline{\underline{y = x^2}}$ ——— (1)(a)

Rewriting (1) . . . $p_y(y) = p(x) \underbrace{\left|\frac{dx}{dy}\right|}$ ——— (2)

$\hookrightarrow$ Jacobian of the transformation

take $\frac{d}{dy}$ .... (1)(a) $\Rightarrow$ $1 = 2x \frac{dx}{dy}$ $\Rightarrow$ $\frac{dx}{dy} = \frac{1}{2x}$

Plugging this in eqn ②, ...

$$p_y(y) = p(x) \left| \frac{1}{2x} \right|$$

$$= \frac{1}{2} \cdot p(x) \frac{1}{|x|}$$

Recall .. $y = x^2$

$x = \pm\sqrt{y} = \sqrt{y}$ [$p(x)$ defined only for $x > 0$]

$$= \frac{1}{2} \cdot p(x) \cdot \frac{1}{\sqrt{y}}$$

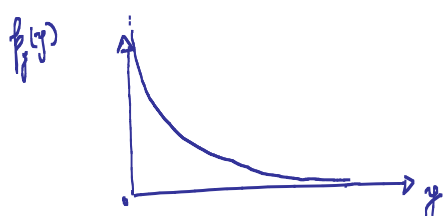
$$= \frac{1}{2} \frac{e^{-\sqrt{y}}}{\sqrt{y}}$$

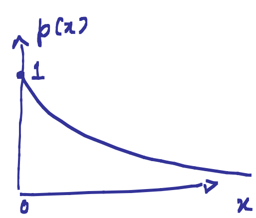
.... $p(x)$ defined only for $x > 0$

$$\boxed{f_y(y) = \frac{e^{-\sqrt{y}}}{2\sqrt{y}}}$$

$$\boxed{p(x) = e^{-x}}$$

$y = x^2$





Check if $f_y(y)$ is normalized.

$$\int_0^{\infty} f_y(y)\, dy = \int_0^{\infty} \frac{e^{-\sqrt{y}}}{2\sqrt{y}}\, dy$$

substitution of $\sqrt{y} = u$

$$\frac{1}{2\sqrt{y}}\, dy = du$$

$y : 0 \rightarrow \infty$

$u : 0 \rightarrow \infty$

$$\int_0^{\infty} e^{-u}\, du$$

$$= -e^{-u} \Big|_0^{\infty}$$

$$= 1$$

"$f_y(y)$ is well behaved"