Key results:

**Discrete random variables.**

$S = \{x_1, x_2, \dots x_N\}$

$$\sum_{i=1}^{N} p(x_i) = 1.$$

$p(x_i)$ are probabilities.

**Continuous random variables**

$S = \{a, b\}$

$a < x < b$ is some continuous variable.

$$\int_a^b p(x)\,dx = 1$$

$p(x)$ is a probability density function

PDF

$p(x)\,dx \equiv$ probability of finding a measurement in the range $[x, x+dx]$

$\downarrow$

$1/x$

#. To obtain average of any function of $x$, say $f(x)$

$x : p(x)$

$$\langle f(x) \rangle = \int_a^b p(x)\, f(x)\, dx$$

if $f(x) = x$

$$\langle x \rangle = \int_a^b p(x)\, x\, dx$$

$\vdots$

$f(x) = x^n$

$$\boxed{\langle f(x) \rangle = \langle x^n \rangle = \int_a^b p(x)\, x^n\, dx \quad \dots \quad \text{" } n^{th} \text{ moment of } x \text{ "}}$$

# Method of generating function :

Generating function is the characteristic function of PDF.

Suppose you have a random variable $x \in [-\infty, \infty]$

Distributed as $p(x)$

To compute $\langle x^m \rangle$, first we compute

Characteristic function $= \tilde{p}(k) = \int_{x=-\infty}^{+\infty} p(x)\, e^{-ikx}\, dx = \langle e^{-ikx} \rangle$ ... $\langle f(x) \rangle = \int_{-\infty}^{\infty} f(x)\, p(x)\, dx$

$$\therefore \tilde{p}(k) = \left\langle \sum_{j=0}^{\infty} \frac{(-ik)^j\, x^j}{j!} \right\rangle = \sum_{j=0}^{\infty} \underbrace{\frac{(-ik)^j}{j!}}_{\text{coefficient}} \langle x^j \rangle$$

To extract $\langle x^m \rangle = \dfrac{\partial^m}{\partial(-ik)^m} \tilde{p}(k) \Big|_{k=0}$ " Hence $\tilde{p}(k)$ is called the generator of moment "

$$\boxed{\langle x^m\rangle = \frac{\partial^m}{\partial(-ik)^m}\tilde{p}(k)\Big|_{k=0}} = \int_{-\infty}^{+\infty} p(x)\, x^m\, dx$$

Now turn to $\ln \tilde{p}(k)$ :

$$\ln \tilde{p}(k) = F(k) = \cancel{F(0)}^{\,0} + (-ik)F'(0) + \frac{(-ik)^2 F''(0)}{2!} + \frac{(-ik)^3 F'''(0)}{3!}$$

.... Expanding in powers of $(-ik)$

$$F(0) = \ln \tilde{p}(0) = \ln 1 = 0$$

$$\ln \tilde{p}(k) = (-ik)F'(0) + \frac{(-ik)^2 F''(0)}{2!} + \frac{(-ik)^3 F'''(0)}{3!} + \dots$$

$$F'(0) = \langle x\rangle_c$$

$$F''(0) = \langle x^2 \rangle_c$$

$$F'''(0) = \langle x^3 \rangle_c$$

$$\vdots$$

$$F^n(0) = \langle x^n \rangle_c$$

$$\ln \tilde{p}(k) = (-ik)\langle x \rangle_c + \frac{(-ik)^2 \langle x^2 \rangle_c}{2!} + \cdots$$

$$= \sum_{j=1}^{\infty} \frac{(-ik)^j \langle x^j \rangle_c}{j!}$$

Extracting $$\boxed{\langle x^m \rangle_c = \frac{\partial^m}{\partial (-ik)^m} \ln \tilde{p}(k) \Big|_{k=0}}$$

" $\ln \tilde{p}(k)$ is the generator of cumulants "

Relation between moments & cumulants.

$$\langle x \rangle_c = \frac{\partial}{\partial(-ik)} \ln \tilde{p}(k) \Big|_{k=0} = \frac{1}{\tilde{p}(k)} \cdot \frac{\partial \tilde{p}(k)}{\partial(-ik)} \Big|_{k=0} = \frac{\partial}{\partial(-ik)} \tilde{p}(k) \Big|_{k=0} = \langle x \rangle$$

$$\boxed{\langle x \rangle_c = \langle x \rangle}$$

$$\langle x^2 \rangle_c = \frac{\partial^2}{\partial(-ik)^2} \ln \tilde{p}(k) \Big|_{k=0} = \frac{\partial}{\partial(-ik)} \frac{1}{\tilde{p}(k)} \frac{\partial \tilde{p}(k)}{\partial(-ik)} \Big|_{k=0}$$

$$= \frac{-1}{\tilde{p}(k)^2} \left( \frac{\partial \tilde{p}(k)}{\partial(-ik)} \right)^2 \Big|_{k=0} + \frac{1}{\tilde{p}(k)} \frac{\partial^2}{\partial(-ik)^2} \tilde{p}(k) \Big|_{k=0}$$

$$= -\langle x \rangle^2 + \langle x^2 \rangle \quad \ldots\ldots \because \tilde{p}(0) = 1.$$

$$= \langle x^2 \rangle - \langle x \rangle^2$$

$$= \sigma^2 \quad \text{"Variance"}$$

$$\langle x^2 \rangle_c = \sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

Similarly,
$$\langle x^3\rangle_c = \frac{\partial^3}{\partial(-ik)^3} \ln \tilde{p}(k) \Big|_{k=0}$$
$$= \langle x^3\rangle - 3\langle x\rangle\langle x^2\rangle + 2\langle x\rangle^3 \quad \ldots\ldots \text{ "Skewness"}$$

Finally,
$$\langle x^4\rangle_c = \frac{\partial^4}{\partial(-ik)^4} \ln \tilde{p}(k) \Big|_{k=0}$$
$$\vdots$$
$$= \langle x^4\rangle - 4\langle x^3\rangle\langle x\rangle - 3\langle x^2\rangle^2 + 12\langle x^2\rangle\langle x\rangle^2 - 6\langle x\rangle^4 \quad \text{"kurtosis"}$$

Summarize:

Moment generating function : $\tilde{p}(k)$
$$\langle x^m\rangle = \frac{\partial^m}{\partial(-ik)^m} \tilde{p}(k) \Big|_{k=0}$$

Cumulant generating function : $\ln \tilde{p}(k)$
$$\langle x^m\rangle_c = \frac{\partial^m}{\partial(-ik)^m} \ln \tilde{p}(k) \Big|_{k=0}$$

Diagrammatic connect moments & cumulants:

Logic: Unconnected dots $\longrightarrow$ moments

Connected dots $\longrightarrow$ cumulants

$$\langle x \rangle = \langle x \rangle_c$$

$$\langle x^2 \rangle = \langle x \rangle_c^2 + \langle x^2 \rangle_c$$

$$\langle x^3 \rangle = \langle x \rangle_c^3 + 3\langle x^2 \rangle_c \langle x \rangle_c + \langle x^3 \rangle_c$$

$$\langle x^4 \rangle = \langle x \rangle_c^4 + 4\langle x \rangle_c \langle x^3 \rangle_c + 6\langle x^2 \rangle_c \langle x \rangle_c^2 + 3\langle x^2 \rangle_c^2 + \langle x^4 \rangle_c$$