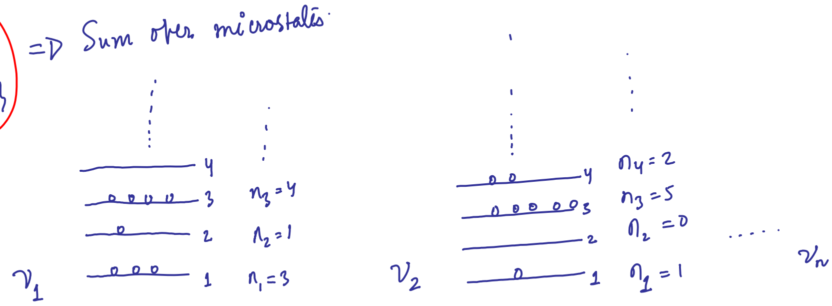
Statistics of Fermions & Bosons.

Bosons: Integer spin particles

No restriction on n_j "No. of particles in ϵ_j"

Partition function : $Z(N,V,T) = \sum_{\{n_i\}} e^{-\beta H(\{n_i\})}$

$\sum_{\{n_i\}}$ $\Rightarrow$ Sum over microstates.



$\{n_1=3, n_2=1, n_3=4, \ldots\}$ $\{n_1=1, n_2=0, n_3=5, n_4=2, \ldots\}$

$$Z(N,V,T) = \sum_{\{n_i\}} e^{-\beta H(\{n_i\})}$$

Hamiltonian $H(\{n_i\}) = \sum_i \epsilon_i n_i$

$$Z(N,V,T) = \sum_{\{n_i\}} e^{-\beta \sum_j \epsilon_j n_j}$$

$$= \sum_{\{n_i\}} e^{-\beta \epsilon_1 n_1} \cdot e^{-\beta \epsilon_2 n_2} \cdots\cdots$$

$$= \prod_j \sum_{n_j=0}^{N} e^{-\beta \epsilon_j n_j}$$

$\Downarrow$

Restricted to follow $\sum_i n_i = N$

This constrained summation is very hard to perform

Solution : Shift to Grand Canonical Ensemble (μ, V, T)

$$\mathcal{Z}(\mu, V, T) = \sum_{\{n_i\}} e^{-\beta \mathcal{H}(\{n_i\})}$$

$$\mathcal{H}(\{n_i\}) = \sum_j n_j \epsilon_j - n_j \mu \quad \cdots\cdots \quad \sum_j n_j = N$$

$$\mathcal{Z}(\mu, V, T) = \sum_{\{n_i\}} e^{-\beta \sum_j n_j (\epsilon_j - \mu)}$$

$$= \prod_j \sum_{n_j = 0}^{\infty} e^{-\beta n_j (\epsilon_j - \mu)}$$

$\Downarrow$

Unrestricted sum "Performed easily"

$$= \prod_j \underbrace{\left[1 + e^{-\beta(\epsilon_j - \mu)} + e^{-2\beta(\epsilon_j - \mu)} + \cdots \right]}_{\substack{\text{Geometric series} \\ r = e^{-\beta(\epsilon_j - \mu)}}}$$

$$\mathcal{Z}(\mu, V, T) = \prod_j \left[\frac{1}{1 - e^{-\beta(\epsilon_j - \mu)}} \right] \longleftarrow \text{converges} \qquad (1)$$

$\Downarrow$

Prefered in Q.M

$$\therefore \ln \mathcal{Z}(\mu, V, T) = -\sum_j \ln \left[1 - e^{-\beta(\epsilon_j - \mu)} \right] \qquad (2)$$

Deriving statistics is easy:

Probability of $\nu = \{n_i\}$: $p(\nu) = p(\{n_i\}) = \dfrac{e^{-\beta \mathcal{H}(\{n_i\})}}{\mathcal{Z}(\mu, V, T)}$

$\therefore$ $\langle n_j \rangle$ = Average occupancy of j^{th} level

$$= \sum_{\{n_i\}} n_j \, p(\{n_i\})$$

$$= \frac{\sum_{\{n_i\}} n_j \, e^{-\beta \mathcal{H}(\{n_i\})}}{\mathcal{Z}(\mu, V, T)} = \frac{\sum_{\{n_i\}} n_j \, e^{-\beta \sum_p (n_p \epsilon_p - n_p \mu)}}{\mathcal{Z}(\mu, V, T)}$$

$$= \frac{\sum_{\{n_i\}} \frac{\partial}{\partial(-\beta \epsilon_j)} e^{-\beta \sum_p n_p(\epsilon_p - \mu)}}{\mathcal{Z}(\mu, V, T)}$$

$$= \frac{\frac{\partial}{\partial(-\beta \epsilon_j)} \underbrace{\sum_{\{n_i\}} e^{-\beta \sum_p n_p(\epsilon_p - \mu)}}_{\mathcal{Z}(\mu, V, T)}}{\mathcal{Z}(\mu, V, T)}$$

$$= \frac{1}{\mathcal{Z}} \frac{\partial}{\partial(-\beta \epsilon_j)} \mathcal{Z}$$

$$\langle n_j \rangle = \frac{\partial}{\partial(-\beta \epsilon_j)} \ln \mathcal{Z} \quad \text{——— } (3)$$

Recalling, $\ln \mathcal{Z} = -\sum_j \ln\left[1 - e^{-\beta(\epsilon_j - \mu)}\right]$

Eqⁿ (3) becomes,

$$\langle n_j \rangle = \frac{e^{-\beta(\epsilon_j - \mu)}}{1 - e^{-\beta(\epsilon_j-\mu)}} = \frac{1}{e^{\beta(\epsilon_j-\mu)} - 1}$$

$$\langle n_j \rangle_{B.E.} = \frac{1}{e^{\beta(\epsilon_j-\mu)} - 1}$$

"Bose-Einstein Statistics"

Fermions "Half integer spins"

Odd parity Wave function

Pauli's exclusion principle

Starting from...

$$\mathcal{Z}(\mu, V, T) = \sum_{\{n_i\}} e^{-\beta \sum_j n_j(\epsilon_j - \mu)}$$

$$= \prod_j \sum_{n_j = 0,1} e^{-\beta n_j(\epsilon_j - \mu)}$$

.... Fermions: $n_j = 0, 1$

$$\mathcal{Z}(\mu, V, T) = \prod_j \left[1 + e^{-\beta(\epsilon_j - \mu)}\right]$$

$$\ln \mathcal{Z}(\mu, V, T) = \sum_j \ln\left[1 + e^{-\beta(\epsilon_j - \mu)}\right]$$

$$\langle n_j \rangle = \sum_{\{n_i\}} n_j P(\{n_i\}) = \frac{\sum_{\{n_i\}} n_j e^{-\beta H(\{n_i\})}}{\mathcal{Z}(\mu, V, T)}$$

$$= \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial(-\beta\epsilon_j)} = \frac{\partial}{\partial(-\beta\epsilon_j)} \ln \mathcal{Z}$$

Giving... $$\langle n_j \rangle_{F.D.} = \frac{e^{-\beta(\epsilon_j-\mu)}}{1 + e^{-\beta(\epsilon_j-\mu)}} = \frac{1}{e^{\beta(\epsilon_j-\mu)} + 1}$$

"Fermi-Dirac Statistics"

Combining both.... $\langle n_j \rangle_{\pm} = \dfrac{1}{e^{\beta(\epsilon_j - \mu)} \pm 1}$

$+$: FD

$-$: BE