Mathematical prerequisites:

1. Gaussian Integral : $\displaystyle\int_{x=-\infty}^{+\infty} e^{-ax^2}\, dx = I$

2. Sadde point approximation: $S = \sum_{i=1}^{I} \underbrace{e^{N x_i}}_{\text{positive}}$

$$\approx e^{N x_{max}} \quad \text{if } N \to \infty$$

3. Fourier transform : If, $p(x) = \dfrac{a}{\pi(a^2+x^2)}$, a is const.

Compute $\tilde{p}(k) = \displaystyle\int_{x=-\infty}^{+\infty} p(x)\, e^{ikx}\, dx$

4. Laplace transform: If $f(x) = x^n$

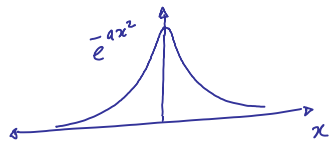
$$\mathcal{L}[f(x)] = \int_{x=0}^{\infty} e^{-sx} x^n\, dx = F(s,n)$$

$$= \frac{n!}{s^{n+1}}$$

$$n! = F(s,n)\Big|_{s=1} = \Gamma(n+1)$$

$$\Gamma(n) = (n-1)!$$

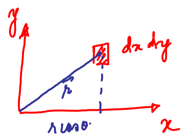
Solution : ① Gaussian Integral: $I = \int_{x=-\infty}^{+\infty} e^{-ax^2} dx$



$$I^2 = \underbrace{\int_{x=-\infty}^{+\infty} e^{-ax^2} dx}_{\text{independent of } x} \underbrace{\int_{y=-\infty}^{+\infty} e^{-ay^2} dy}_{\text{independent of } y}$$

$$= \int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{+\infty} e^{-a(x^2+y^2)} dx\, dy$$

"Area integral"



$$dx\,dy = \underbrace{dr}_{dl_r} \underbrace{(r d\theta)}_{dl_\theta}$$ //

$$x^2+y^2 = r^2$$

$$I^2 = \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} e^{-ar^2} r\, dr\, d\theta$$

$$= 2\pi \int_{r=0}^{\infty} e^{-ar^2} r\, dr$$

.... $r^2 = u/a \Rightarrow 2r\,dr = \frac{1}{a} \cdot du$

$$I^2 = \frac{2\pi}{2a}\int_0^{\infty} e^{-u}\,du = \frac{\pi}{a}\int_0^{\infty} e^{-u}\,du = \frac{\pi}{a}\cdot\frac{e^{-u}}{-1}\Big|_0^{\infty}$$

$$= \frac{\pi}{a}$$

$$\Rightarrow I = \sqrt{\frac{\pi}{a}} = \int_{x=-\infty}^{+\infty} e^{-ax^2}\,dx$$

② Saddle point approximation: $S = \sum_{i=1}^{T} e^{Nf(x_i)}$

$f(x) = x$
$S = \sum_{i=1}^{T} e^{Nx_i}$

$f(x) = x^2$
$S = \sum_{i=1}^{T} e^{Nx_i^2}$

$$S \approx e^{Nf^{max}(x^*)} \quad \text{if } N \to \infty$$

$$S = e^{Nf(x_1)} + e^{Nf(x_2)} + \dots\dots \text{ T terms} \dots e^{Nf(x_T)}$$

$$> e^{Nf^{max}(x^*)}$$

$$< T e^{Nf^{max}(x^*)}$$

This gives $\rightarrow$ $$e^{Nf^{max}(x^*)} < S < T e^{Nf^{max}(x^*)}$$

Taking logarithm (base e) :

$$Nf^{max}(x^*) < \ln S < \ln T + Nf^{max}(x^*)$$

Dividing throughout by N:

$$f^{max}(x^*) < \frac{\ln S}{N} < \frac{\ln T}{N} + f^{max}(x^*)$$

Taking $T = N^a$ for eg: $T = 100$

$N = 100 \Rightarrow a = 1$

$N = 1000 \Rightarrow a = 2/3$

$$f^{max}(x^*) < \frac{\ln S}{N} < \frac{a \ln N}{N} + f^{max}(x^*)$$

For large N: $\frac{\ln N}{N} \ll 1$

$N = 10^2$

$\frac{\ln N}{N} \sim \frac{2}{100}$

$N = 10^8$, $\frac{\ln N}{N} \sim \frac{8}{10^8}$

Give us:
as $N \to \infty$

$$f^{max}(x^*) < \frac{\ln S}{N} < f^{max}(x^*)$$

Only possible: $\frac{\ln S}{N} \simeq f^{max}(x^*)$

$$S = e^{N f^{max}(x^*)}$$

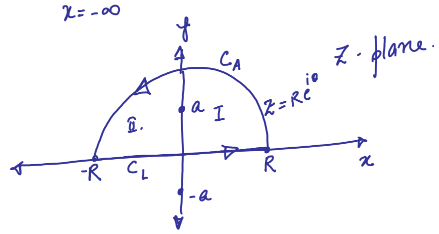
③ Fourier transform: $p(x) = \frac{a}{\pi(a^2+x^2)}$

$$\tilde{p}(k) = \int_{x=-\infty}^{+\infty} p(x)\, e^{ikx}\, dx$$

$$p(x) = \frac{1}{2\pi} \int_{k=-\infty}^{+\infty} \tilde{p}(k)\, e^{-ikx}\, dk$$

$$\tilde{p}(k) = \int_{x=-\infty}^{+\infty} \frac{a}{\pi(a^2+x^2)} e^{ikx}\, dx = \frac{a}{\pi} \cdot \int_{x=-\infty}^{+\infty} \frac{e^{ikx}}{(a^2+x^2)}\, dx \quad \text{—— (1)}$$

$$I_z = \oint_C \frac{e^{ikz}}{(a^2+z^2)}\, dz.$$



The integrand $\frac{e^{ikz}}{(a^2+z^2)} = \frac{e^{ikz}}{(z+ia)(z-ia)}$

$$I_z = 2\pi i \cdot \mathrm{Res}\,[z=ia] = 2\pi i \cdot \frac{e^{ik\,ia}}{(2ia)} \quad \because \ \mathrm{Res}\,[z=z_0] = \lim_{z\to z_0} (z-z_0) f(z)$$

$$= \frac{\pi}{a} \cdot e^{-ka}$$

$$I_z = \oint_C \frac{e^{ikz}}{(z^2+a^2)}\, dz = \underbrace{\int_{x=-R}^{+R} \frac{e^{ikx}}{(x^2+a^2)}\, dx}_{C_L} + \underbrace{\int_{\theta=0}^{\pi} \frac{e^{ikRe^{i\theta}}}{(R^2 e^{2i\theta}+a^2)} Rie^{i\theta}\, d\theta}_{C_A}$$

$$\because \ z = Re^{i\theta}, \quad dz = Rie^{i\theta} d\theta$$

Now $\therefore$

$$\underbrace{\int_{\theta=0}^{\pi} \frac{e^{ikRe^{i\theta}}}{(R^2 e^{2i\theta}+a^2)} Rie^{i\theta}\, d\theta}_{C_A} = \cancel{\int_{\theta=0}^{\pi} \frac{e^{ikR\cos\theta} \cdot e^{-kR\sin\theta}\, iR e^{i\theta}\, d\theta}{(R^2 e^{2i\theta}+a^2)}} \to 0$$

for $k>0$, $\sin\theta > 0$

as $R \to \infty$

I_z as $R \to \infty$ remains $\frac{\pi}{a} e^{-ka}$

$$\lim_{R\to\infty} \int_{-R}^{+R} \frac{e^{ikx}}{(a^2+x^2)}\, dx + \underbrace{\cancel{\int_{\theta=0}^{\pi} \frac{e^{ikz}}{(a^2+z^2)}\, dz}}_{C_A} = \frac{\pi}{a} e^{-ka}$$

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}}{a^2+x^2}\, dx = \frac{\pi}{a} \cdot e^{-|k|a}$$

$$\therefore \tilde{p}(k) = \frac{a}{\pi} \cdot \frac{\pi}{a} \cdot e^{-|k|a} = e^{-|k|a}$$

4.) Laplace transform: $L[f(x)] = \int_{x=0}^{\infty} f(x)\, e^{-sx}\, dx = F(s)$

For. eg. $f(x) = x^n$: $L[f(x)] = L[x^n] = F(s, n)$

$$= \int_{x=0}^{\infty} x^n e^{-sx} dx$$

$$L[x^n] = \int_{x=0}^{\infty} \underset{u}{x^n}\, \underset{v}{e^{-sx}}\, dx$$

$$= \cancel{\left. \frac{x^n e^{-sx}}{-s} \right|_0^{\infty}} + \frac{n}{s} \int_{x=0}^{\infty} x^{n-1} e^{-sx} dx$$

$$= \left(\frac{n}{s}\right) \mathcal{L}[x^{n-1}]$$

$$= \left(\frac{n}{s}\right)\left(\frac{n-1}{s}\right) \mathcal{L}[x^{n-2}]$$

$$\vdots$$

$$= \left(\frac{n}{s}\right)\left(\frac{n-1}{s}\right)\left(\frac{n-2}{s}\right) \cdot \frac{1}{s} \cdot \mathcal{L}[x^0]$$

$$= \frac{n!}{s^n} \cdot \mathcal{L}[1]$$

$$\mathcal{L}\left[x^{n}\right] = \frac{n!}{s^{n+1}} = F(s,n)$$

$$\therefore \; n! = F(s,n)\Big|_{s=1} = \Gamma(n+1)$$

Since $n! = n(n-1)! = n\,\Gamma(n) = \Gamma(n+1)$