Binomial distribution:

Interested in observations that have two outcomes (say A or B)

Associated probabilities are p_A and $p_B = 1 - p_A$

Because $p_A + p_B = 1$.

Toss a coin N times, the probability of getting N_A occurrences of event A is -

$$p_N(N_A) = \underbrace{p_A^{N_A}\, p_B^{N-N_A}}_{\text{Single ordering}} \cdot \underbrace{\binom{N}{N_A}}_{\frac{N!}{N_A!\,(N-N_A)!}\ \text{Total orderings!}}$$

$N = 5$
$N_A = 3$
$N_B = 2$

$p_A^3\, p_B^{5-3} \cdot {}^5C_3$

A A A B B
A B A B A
A B B A A
⋮

${}^5C_2 = {}^5C_3$

Characteristic function:

$$\tilde{p}_N(k) = \sum_{N_A=0}^{N} e^{-ikN_A}\, p_A^{N_A}\, p_B^{N-N_A} \binom{N}{N_A}$$

$$= \sum_{N_A=0}^{N} \left(e^{-ik} p_A\right)^{N_A} p_B^{N-N_A} \binom{N}{N_A}$$

$$\boxed{\tilde{p}_N(k) = \left(e^{-ik} p_A + p_B\right)^N}$$

.... Binomial expansion

Cumulant generating function: $\boxed{\ln \tilde{p}_N(k) = N \ln\left(e^{-ik} p_A + p_B\right)}$

$$\langle N_A \rangle = \frac{\partial}{\partial(-ik)} \tilde{p}_N(k)\Big|_{k=0} = N\left(e^{-ik} p_A + p_B\right)^{N-1} p_A e^{-ik}\Big|_{k=0} = N p_A$$

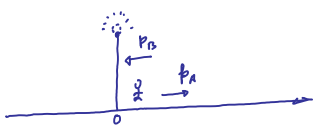
$\because p_A + p_B = 1$

$\langle N_A \rangle = N p_A = N/2 \quad \because p_A = 1/2$ [unbiased coin]

Example: Random walk in one dimension.

Walker is unbiased: $\boxed{p_A = p_B = \frac{1}{2}}$

He takes steps of unit size



What is the probability of finding the walker at location $x = m$, after N steps?

Total steps = N

Steps in forward direction : n_f

Steps in backward direction: $N - n_f$

Net. displacement $m = n_f - (N - n_f) = \underline{2n_f - N}$

$$p_N(\text{loc. } m) = p_N(n_f) = p_f^{n_f}\, p_b^{N-n_f} \binom{N}{n_f}$$

Taking further ... Compute characteristic function.

$$\tilde{p}_N(k) = \left(p_f e^{-ik} + p_B\right)^N = \underline{(e^{-ik}+1)^N \left(\tfrac{1}{2}\right)^N} \quad \because p_f = p_b = 1/2$$

Lets compute the averages:

$$\boxed{\begin{aligned} \langle m\rangle &= \langle 2n_f - N\rangle = 2\langle n_f\rangle - N \\ \langle m^2\rangle &= \langle (2n_f - N)^2\rangle = 4\langle n_f^2\rangle - 4\langle n_f\rangle N + N^2 \\ \langle m^3\rangle &= 8\langle n_f^3\rangle - N^3 - 12N\langle n_f^2\rangle + 6N^2\langle n_f\rangle \\ \langle m^4\rangle &= 16\langle n_f^4\rangle - 32\langle n_f^3\rangle N + 24\langle n_f^2\rangle N^2 - 8\langle n_f\rangle N^3 + N^4 \end{aligned}} \Bigg\} \text{ Moments !}$$

Required quantities are ... $\langle n_f\rangle, \langle n_f^2\rangle, \langle n_f^3\rangle, \langle n_f^4\rangle$

$$\langle n_f\rangle = \frac{\partial}{\partial(-ik)} \tilde{p}_N(k)\Big|_{k=0} = \frac{N}{2}.$$

$$\langle n_f^2\rangle = \frac{\partial^2}{\partial(-ik)^2} \tilde{p}_N(k)\Big|_{k=0} = N^2/4 + N/4$$

$$\langle n_f^3\rangle = \frac{\partial^3}{\partial(-ik)^3} \tilde{p}_N(k)\Big|_{k=0} = N(N^2 + 3N)/8$$

$$\langle n_f^4\rangle = \frac{\partial^4}{\partial(-ik)^4} \tilde{p}_N(k)\Big|_{k=0} = (N^4 + 6N^3 + 3N^2 - 2N)/16$$

$$\tilde{p}_N(k) = \left(p_f e^{-ik} + p_B\right)^N = (e^{-ik}+1)^N \left(\tfrac{1}{2}\right)^N$$

$$\langle n_f\rangle = \frac{\partial}{\partial(-ik)} \left(\tfrac{1}{2}\right)^N (e^{-ik}+1)^N \Big|_{k=0} = \left(\tfrac{1}{2}\right)^N \cdot N \left(e^{-ik}+1\right)^{N-1} \cdot e^{-ik}\Big|_{k=0}$$

$$= \left(\tfrac{1}{2}\right)^N \cdot N \cdot 2^{N-1} = N/2$$

$$\langle n_f^2\rangle = \frac{\partial^2}{\partial(-ik)^2} \left(\tfrac{1}{2}\right)^N (e^{-ik}+1)^N \Big|_{k=0} = \left(\tfrac{1}{2}\right)^N \cdot N \cdot \Big[(N-1)(e^{-ik}+1)^{N-2} \cdot e^{-ik} \cdot e^{-ik} + (e^{-ik}+1)^{N-1} \cdot e^{-ik}\Big]_{k=0}$$

$$= \left(\tfrac{1}{2}\right)^N N \left[(N-1)\, 2^{N-2} + 2^{N-1}\right]$$

$$= N\left[\frac{(N-1)}{4} + \frac{1}{2}\right]$$

$$= N\left[\frac{N-1+2}{4}\right]$$

$$= \frac{N(N+1)}{4}$$

$\vdots$

$$\left.\begin{aligned} \langle m \rangle &= 0 \\ \langle m^2 \rangle &= N \\ \langle m^3 \rangle &= 0 \\ \langle m^4 \rangle &= 3N^2 - 2N \end{aligned}\right\}$$

All moments can then be calculated !