Transformation of Random variables

Suppose x distributed as $p_X(x)$

$y = f(x) \Rightarrow x = f^{-1}(y)$

What is the PDF of y?

Let $p_Y(y)$ be the distribution.

$$p_X(x)\,dx = p_Y(y)\,dy$$

$$p_Y(y) = p_X(x)\left|\frac{dx}{dy}\right| = p_X(x)\,|J|$$

$$\boxed{p_Y(y) = p_X(x)\,|J|}$$

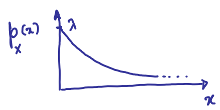
$dx\,dy = J(r,\theta)\,dr\,d\theta$ ← Area magnification

$= r\,dr\,d\theta$

$dx\,dy\,dz = J(r,\theta,\phi)\,dr\,d\theta\,d\phi$

$= r^2\sin\theta\,dr\,d\theta\,d\phi$

Example: $x : p_X(x) = \lambda e^{-\lambda x} \quad x \in [0, \infty]$



If, $y = x^2$ has a PDF $p_Y(y)$,

Conservation of probability, $p_Y(y)\,dy = p_X(x)\,dx$ —— ①

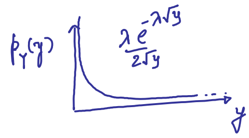
$$p_Y(y) = p_X(x)\left|\frac{dx}{dy}\right|$$

$1 = 2x \cdot \frac{dx}{dy}$

$\frac{dx}{dy} = \frac{1}{2x}$

$$= p_X(x)\,\frac{1}{2|x|}$$

$$= \frac{\lambda e^{-\lambda\sqrt{y}}}{2\sqrt{y}}, \quad x \in [0, \infty]$$



Norm: $$\int_{y=0}^{\infty} p_Y(y)\,dy = \lambda\int_{y=0}^{\infty} \frac{e^{-\lambda\sqrt{y}}}{2\sqrt{y}}\,dy$$

Substitute $\sqrt{y} = u$

$$\frac{1}{2\sqrt{y}} \cdot dy = du$$

$$\int_{y=0}^{\infty} p_Y(y)\,dy = \lambda \int_0^{\infty} e^{-\lambda u}\,du = \lambda \cdot \frac{1}{\lambda} = 1$$

$$p_X(x) = \lambda e^{-\lambda x}$$

$$p_Y(y) = \frac{\lambda}{2\sqrt{y}} e^{-\lambda\sqrt{y}}$$

(b) $x : p_X(x)$

$y : p_Y(y)$

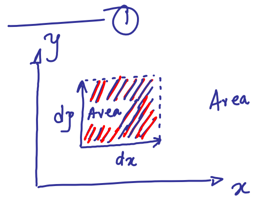
$u = x + y \quad , \quad v = x$

$p_U(u) = ?$

$$p_{XY}(x,y) = p_X(x)\, p_Y(y)$$

Conservation of probability:

$$p_{XY}(x,y)\,dx\,dy = p_{UV}(u,v)\,du\,dv$$



Area = $dx\,dy$

$x = x(u,v) \quad \Rightarrow \quad u = u(x,y)$

$y = y(u,v) \qquad\quad v = v(x,y)$

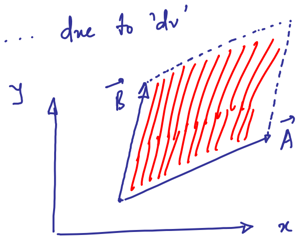
$$dx = \left(\frac{\partial x}{\partial u}\right) du + \left(\frac{\partial x}{\partial v}\right) dv$$

$$dy = \left(\frac{\partial y}{\partial u}\right) du + \left(\frac{\partial y}{\partial v}\right) dv$$

Two displacements due (u,v)

$\vec{A} = \left(\frac{\partial x}{\partial u}\right) du\, \hat{i} + \left(\frac{\partial y}{\partial u}\right) du\, \hat{j}$ .... due to 'du'

$\vec{B} = \left(\frac{\partial x}{\partial v}\right) dv\, \hat{i} + \left(\frac{\partial y}{\partial v}\right) dv\, \hat{j}$ ..... due to 'dv'



$|\vec{A} \times \vec{B}|$

Area of Rectangle = Area of Parallelogram

$dx\,dy = |\vec{A} \times \vec{B}|$

$$= \left| A_x B_y - B_x A_y \right|$$

$$= \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| du\, dv$$

$$dx\,dy = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| du\,dv = \Big| \underbrace{\frac{\partial(x,y)}{\partial(u,v)}}_{J(u,v)} \Big|\, du\,dv$$

$$\left.\begin{aligned} p_{XY}(x,y)\,dx\,dy &= p_{UV}(u,v)\,du\,dv \\ &= p_{XY}(x,y)\,|J|\,du\,dv \end{aligned}\right\}$$

$$p_{UV}(u,v) = p_{XY}(x,y)\,|J| \qquad \text{—— (2)}$$

Suppose $x: p_X(x) = \lambda e^{-\lambda x}$

$y: p_Y(y) = \beta e^{-\beta y}$

$u = x + y$

$v = x$

$$\therefore J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \qquad \text{—— (3)}$$

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(u-y) = 1 \qquad \frac{\partial x}{\partial v} = \frac{\partial x}{\partial x} = 1$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(u-x) = 1 \qquad \frac{\partial y}{\partial v} = \frac{\partial y}{\partial x} = 0$$

$$\therefore J = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$\begin{aligned} p_{UV}(u,v) &= p_{XY}(x,y) \\ &= p_X(x)\,p_Y(y) \\ &= \lambda\beta\, e^{-\lambda x}\, e^{-\beta y} \\ &= \lambda\beta\, e^{-\lambda v} \cdot e^{-\beta(u-v)} \end{aligned}$$

$u = x + y$

$v = x$

$x \in [0, \infty]$

$y \in [0, \infty]$

$$p_U(u) = \int_{v=0}^{u} p_{UV}(u,v)\,dv$$

$$= \int_{v=0}^{v=u} \lambda\beta\, e^{-\lambda v}\, e^{-\beta(u-v)}\, dv$$

$$= \lambda\beta\, e^{-\beta u} \int_{v=0}^{u} e^{-v(\lambda-\beta)}\, dv$$

$$= \lambda\beta\, e^{-\beta u} \left. \frac{e^{-v(\lambda-\beta)}}{(\beta-\lambda)} \right|_{v=0}^{v=u}$$

$$= \frac{\lambda\beta}{(\beta-\lambda)} \cdot e^{-\beta u} \left[e^{-u(\lambda-\beta)} - 1 \right]$$

$$p_U(u) = \frac{\lambda\beta}{(\beta-\lambda)} \left[e^{-\lambda u} - e^{-\beta u} \right] \qquad \underline{\underline{u = x+y}}$$

$$\left.\begin{aligned} u &= xy \\ v &= y \end{aligned}\right\} \quad p_U(u) = ?$$

can thus be calculated easily!