$x : p(x)$

Characteristic function $= \tilde{p}(k) = \int_{x=-\infty}^{+\infty} p(x)\, e^{-ikx}\, dx$

$$\boxed{\langle x^m \rangle = \frac{\partial^m}{\partial(-ik)^m} \tilde{p}(k) \Big|_{k=0}} = \int_{x=-\infty}^{+\infty} p(x)\, x^m\, dx$$

Cumulant generating function: $\ln \tilde{p}(k) = \sum_{j=1}^{\infty} \frac{(-ik)^j}{j!} \langle x^j \rangle_c$

$$\boxed{\langle x^m \rangle_c = \frac{\partial^m}{\partial(-ik)^m} \ln \tilde{p}(k) \Big|_{k=0}}$$

$\langle x \rangle = \cdot = \langle x_c \rangle$

$\langle x^2 \rangle = \underset{\langle x \rangle^2}{\cdot\,\cdot} + \underset{\langle x^2 \rangle_c}{\odot} = \langle x \rangle_c^2 + \langle x^2 \rangle_c$

$\langle x^3 \rangle = \underset{\langle x \rangle^3}{\therefore} + \underset{3\langle x^2 \rangle_c \langle x \rangle}{3\, \odot\!\cdot} + \underset{\langle x^3 \rangle_c}{\odot} = \langle x \rangle_c^3 + 3 \langle x^2 \rangle_c \langle x \rangle_c + \langle x^3 \rangle_c \qquad \because \langle x \rangle = \langle x \rangle_c$

$\vdots$

Very important distributions in Stat Mech:

(a) Gaussian distribution: $p(x) = N e^{-ax^2}$

N to be determined by normalization

$$\int_{-\infty}^{+\infty} p(x)\, dx = 1 = \int_{-\infty}^{+\infty} N e^{-ax^2} dx = N \int_{-\infty}^{+\infty} e^{-ax^2} dx = N\sqrt{\frac{\pi}{a}}$$

$\Rightarrow N = \sqrt{\frac{a}{\pi}}$

$p(x) = \sqrt{\frac{a}{\pi}}\, e^{-ax^2}$

Substitute $a = 1/2\sigma^2$ (say)

$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-x^2/2\sigma^2}$ ... Normalized Gaussian

$\langle x \rangle = 0$

Check this: $\langle x \rangle = \int_{-\infty}^{+\infty} \underbrace{x\, p(x)}_{\text{odd } f^n} dx = 0$

$I = \int_{-\infty}^{+\infty} e^{-ax^2} dx$

$\therefore I^2 = \iint_{-\infty}^{+\infty} e^{-a(x^2+y^2)} dx\, dy$

$x^2 + y^2 = r^2$

$dx\,dy = dr\, r\, d\theta$

$I^2 = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-ar^2} r\, dr\, d\theta$

$= \frac{2\pi}{2a} \int_{u=0}^{\infty} e^{-u}\, du \qquad ar^2 = u, \; a 2r dr = du$

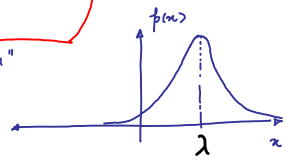
$= \frac{\pi}{a} (-e^{-u}) \Big|_0^{\infty} = \frac{\pi}{a}$

$\Rightarrow \boxed{I = \sqrt{\pi/a}}$

$\langle x^{2n+1} \rangle = 0 \qquad n = 0, 1, \ldots$

For a finite mean: $\boxed{p(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \, e^{-(x-\lambda)^2/(2\sigma^2)}}$

Distribution has $\langle x \rangle = \lambda$. "Mean"



<u>Characteristic function:</u> $\tilde{p}(k) = \displaystyle\int_{x=-\infty}^{+\infty} e^{-ikx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\lambda)^2/(2\sigma^2)} \, dx$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{x=-\infty}^{+\infty} e^{-(x-\lambda+\sigma^2 ik)^2/2\sigma^2} \cdot e^{-ik\lambda} \cdot e^{-k^2\sigma^2/2} \, dx$$

$$= \frac{e^{-ik\lambda}}{\sqrt{2\pi\sigma^2}} \cdot e^{-k^2\sigma^2/2} \int_{x=-\infty}^{+\infty} e^{-(x-\lambda+\sigma^2 ik)^2/2\sigma^2} \, dx$$

$x - \lambda + \sigma^2 ik = u$

$$= \frac{e^{-ik\lambda - k^2\sigma^2/2}}{\sqrt{2\pi\sigma^2}} \cdot \underbrace{\int_{u=-\infty}^{+\infty} e^{-u^2/2\sigma^2} \, du}_{\sqrt{\frac{\pi}{(1/2\sigma^2)}}}$$

$$\tilde{p}(k) = e^{-ik\lambda - k^2\sigma^2/2}$$

Cumulant generator: $\ln \tilde{p}(k) = -ik\lambda - k^2\sigma^2/2 = -ik\lambda + (-ik)^2\sigma^2/2$

$\langle x \rangle_c = \dfrac{\partial}{\partial(-ik)} \ln \tilde{p}(k) \Big|_{k=0} = \lambda$ "Mean"

$\langle x^2 \rangle_c = \dfrac{\partial^2}{\partial(-ik)^2} \ln \tilde{p}(k) \Big|_{k=0} = \sigma^2$ "Variance"

$\langle x^3 \rangle_c = 0$

$\langle x^4 \rangle_c = 0$

$\vdots$

$\langle x^m \rangle_c = 0$

Gaussian PDF is completely specified by first two cumulants - $\langle x \rangle = \lambda$ & $\langle x^2 \rangle_c = \sigma^2$

Some moments of $p(x)$: Gaussian:

$$\langle x \rangle = \bullet = \langle x \rangle_c = \lambda$$

$$\langle x^2 \rangle = \bullet\,\bullet + \bullet\bullet = \lambda^2 + \sigma^2$$

$$\langle x^3 \rangle = \therefore + 3\,\bullet\bullet\;\bullet + \therefore = \lambda^3 + 3\sigma^2\lambda + 0 = \lambda^3 + 3\sigma^2\lambda$$

$$\langle x^4 \rangle = ::\; + 4\,\therefore\;\bullet + 6\,\bullet\bullet\;\bullet\,\bullet + 3\,\bullet\bullet\;\bullet\bullet + :: = \lambda^4 + 4(0)\lambda + 6\sigma^2\lambda^2 + 3\sigma^4 + 0$$

$$= \lambda^4 + 6\sigma^2\lambda^2 + 3\sigma^4$$

Suppose you are given:

$$\langle x \rangle_c = \lambda$$

$$\langle x^2 \rangle_c = \sigma^2$$

$$\ln \tilde{p}(k) = \sum_{j=1}^{N} \frac{(-ik)^j \langle x^j \rangle_c}{j!} = (-ik)\lambda + \frac{(-ik)^2}{2!}\sigma^2 = -ik\lambda - \frac{k^2\sigma^2}{2}$$

$$\tilde{p}(k) = e^{-ik\lambda - k^2\sigma^2/2}$$

$$p(x) = \frac{1}{2\pi}\int_{k=-\infty}^{+\infty} e^{ikx}\,\underbrace{e^{-ik\lambda - k^2\sigma^2/2}}_{\tilde{p}(k)}\,dk$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\lambda)^2/2\sigma^2}$$

$\Longrightarrow$ Recovered Gaussian PDF.