Correspondence between Quantum Statistics & Classical statistics.

Recall, $\ln \mathcal{Z}(\mu, V, T) = \pm \sum_j \ln\left(1 \pm e^{\beta(\mu - \epsilon_j)}\right)$ —— (1)

$+$: Fermi Statistics

$-$: Bose statistics

Classical limit: $n\lambda^3 \ll 1$

"low n & high T"

λ : De-Broglie wavelength

$= \dfrac{h}{\sqrt{2\pi k_B T m}}$

Quantum limit: $n\lambda^3 \gg 1$

"high n & low T"

Applying classical limit $(n\lambda^3 \ll 1)$, $\mathcal{Z}(\mu, V, T) \longrightarrow Z(N, V, T)$ ideal gas

$Z(N,V,T) \to \dfrac{V^N}{N!\, h^{3N}} \left(\dfrac{2\pi m}{\beta}\right)^{3N/2}$

Occupation number average:

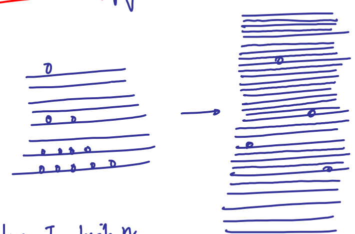
$$\langle n_j \rangle = \frac{1}{e^{\beta(\epsilon_j - \mu)} \pm 1}$$

$+$: FD

$-$: BE

Under high T & low n : Most states become empty

$\underbrace{\text{low } n}$ $\downarrow$ high Volume



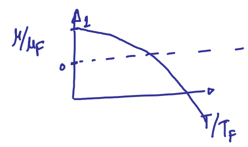
$$\langle n_j \rangle \approx \frac{1}{e^{\beta(\epsilon_j - \mu)} \pm 1} \ll 1$$

$$e^{\beta(\epsilon_j-\mu)} \pm 1 \gg 1$$

$$e^{\beta(\epsilon_j-\mu)} \gg 1$$

$$e^{\beta\epsilon_j} \cdot e^{-\beta\mu} \gg 1$$

$$-\beta\mu \gg 1$$ (β low)
(or T high)



N = Total no. of particles

$$= \sum_j \langle n_j \rangle$$

$$\approx \sum_j \frac{1}{e^{\beta(\epsilon_j-\mu)}}$$ (High T, low n)
Classical limit

$$\approx \sum_j e^{-\beta(\epsilon_j-\mu)}$$

$$\approx e^{\beta\mu} \sum_j e^{-\beta\epsilon_j}$$

$$\therefore \quad e^{\beta\mu} = \frac{N}{\sum_j e^{-\beta\epsilon_j}}, \quad \text{implying} \quad \beta\mu = \ln N - \ln\left(\sum_j e^{-\beta\epsilon_j}\right) \qquad (2)$$

Recalling from Classical S.M.

$$\ln Z(N,V,T) = -\beta F = -\beta(E-TS)$$

$$\ln \mathcal{Z}(\mu,V,T) = -\beta G = -\beta(E-TS-\mu N)$$

$$= \ln Z(N,V,T) + \beta\mu N$$

Rearranging... $\ln Z(N,V,T) = \ln \mathcal{Z}(\mu,V,T) - \beta\mu N$

$$\therefore \ln Z(N,V,T) = \pm \sum_j \ln\Big(1 \pm \underbrace{e^{\beta(\mu-\epsilon_j)}}_{\text{very small}}\Big) - \beta\mu N$$

Since $-\beta\mu \gg 1$ (Classical limit)

$$e^{\beta(\mu-\epsilon_j)} \ll 1$$

$$\therefore \ln\left(1 \pm e^{\beta(\mu-\epsilon_j)}\right) \approx \pm e^{\beta(\mu-\epsilon_j)} + O\left[\left(\beta(\mu-\epsilon_j)\right)^2\right]$$

$\ln(1\pm x) \approx \pm x - \frac{x^2}{2} + O(x^3)$

$x \ll 1$

$\therefore \ln(1\pm x) \approx \pm x + \cancel{O(x^2)}_0$

$$\ln Z(N,V,T) \approx \sum_j e^{\beta(\mu-\epsilon_j)} - \beta\mu N \quad \text{— (3)}$$

$$\approx \underbrace{\sum_j e^{\beta(\mu-\epsilon_j)}}_{N \text{ (Refer previous discussion)}} - N\left[\ln N - \ln\Big(\sum_j e^{-\beta\epsilon_j}\Big)\right]$$

$$\approx N - N\ln N + N\ln\Big(\sum_j e^{-\beta\epsilon_j}\Big)$$

$$\approx -\ln N! + \ln\Big(\sum_j e^{-\beta\epsilon_j}\Big)^N \quad \ldots\ldots \quad N\ln x = \ln x^N$$

$$\ln Z(N,V,T) \approx \ln\left[\frac{\Big(\sum_j e^{-\beta\epsilon_j}\Big)^N}{N!}\right] \quad \ldots\ldots \quad \log A - \log B = \log(A/B)$$

$$Z(N,V,T) \approx \Big(\sum_j e^{-\beta\epsilon_j}\Big)^N \Big/ N! = Z_1^N / N!$$

$\because Z_1$ = Single particle partition function $= \sum_j e^{-\beta\epsilon_j}$

We must determine Z_1 or Z!

$N!$ = In-distinguishability

Go from $\sum_j \xrightarrow{\text{Thermodynamic limit}} \int g(k)\,dk$

$$g(k) = \frac{d^3k}{(\pi/L)^3} \cdot \underbrace{\frac{1}{8}}$$

Avoid negative quantum no.s.

$$Z(N,V,T) = \frac{1}{N!}\left[\frac{V}{8\pi^3}\underbrace{\iiint_{k_x\,k_y\,k_z}}_{-\infty}^{+\infty} e^{-\beta\frac{\hbar^2k^2}{2m}}\,dk_x\,dk_y\,dk_z\right]^{N} \quad \ldots\ldots \quad \epsilon_j = \frac{\hbar^2 k_j^2}{2m}$$

$$= \frac{1}{N!}\left[\frac{V}{(2\pi)^3}\underbrace{\iiint_{k_x\,k_y\,k_z}}_{-\infty}^{+\infty} e^{-\frac{\beta\hbar^2}{2m}(k_x^2+k_y^2+k_z^2)}\,dk_x\,dk_y\,dk_z\right]^{N}$$

$$= \frac{1}{N!}\left[\frac{V^N}{(2\pi)^{3N}}\left(\frac{\pi\,2m}{\beta\hbar^2}\right)^{3N/2}\right] \quad \ldots\ldots \quad \left(\int_{-\infty}^{+\infty} e^{-ax^2}dx\right)^{3N} = \left(\frac{\pi}{a}\right)^{3N/2}$$

$$= \frac{1}{N!}\cdot V^N\left(\frac{2\pi m}{(2\pi)^2\beta\hbar^2}\right)^{3N/2}$$

$$= \frac{1}{N!}\,V^N\left(\frac{2\pi m}{\beta h^2}\right)^{3N/2} \qquad \hbar = h/2\pi$$

$$Z(N,V,T) = \frac{1}{N!}\cdot\frac{1}{h^{3N}}\cdot\left(\frac{2\pi m}{\beta}\right)^{3N/2}\cdot V^N \quad \text{"Ideal gas"}$$

$$= e^{-\beta F}$$

factor of in-distinguishability

Planck's constant.