<u>Central Limit Theorem</u>

Suppose x & y are two random variables

$x \downarrow p(x)$, $y \downarrow q(y)$ be their respective PDF's

You compute sum $z = x + y$

Q. How is the random variable z distributed ?

A. $p(x)\,dx \equiv$ probability of finding x in range $[x, x+dx]$

$q(y)\,dy \equiv$ " " " y " " $[y, y+dy]$

$\phi(z)\,dz \equiv$ probability of finding z in range $[z, z+dz]$

$\cancel{=}\; p(x)\, q(y)\, dx\, dy$

$z = 5$: $x = 2,\ y = 3$ $\quad p(x_0)\, q(y_0)\, dx_0\, dy_0 \Big|_{x_0 = 2,\, y_0 = 3}$

$x = 1,\ y = 4$ $\quad p(x_1)\, q(y_1)\, dx_1\, dy_1 \Big|_{x_1 = 1,\, y_1 = 4}$

⋮

infinite no. of ways

Instead of chasing $\phi(z)$ directly,

I chase the characteristic function $\tilde{\phi}(k)$

$$\tilde{\phi}(k) = \langle e^{-ikz} \rangle = \int_{x=x_{min}}^{x_{max}} \int_{y=y_{min}}^{y_{max}} e^{-ikz} \cdot p(x)\, q(y)\, dx\, dy$$

$$= \int_{x=x_{min}}^{x_{max}} e^{-ikx}\, p(x)\, dx \int_{y=y_{min}}^{y_{max}} e^{-iky}\, q(y)\, dy \quad ,\ldots.\ z = x+y$$

$$= \tilde{p}(k)\ \tilde{q}(k)$$

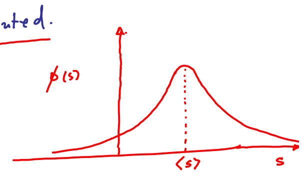
Because x, y are uncorrelated, $\boxed{\tilde{\phi}_z(k) = \tilde{p}_x(k)\ \tilde{q}_y(k)}$

Central Limit theorem: If you sample x from arbitrary PDF $p(x)$
- ↳ Gaussian
- ↳ Exponential
- ↳ Lorentz.
- ↳ flat.
- ⋮

$$s = (x_1 + x_2 + x_3 + x_4 \cdots + x_N)/N \quad \text{—— (1)}$$

since x_i are random, s is also random variable.

If $N \gg 1$, then s is <u>normally distributed</u>.



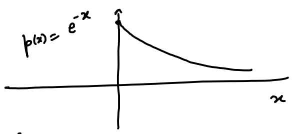
We want $p(s)$!

We will chase $\tilde{\phi}(k) = \int \cdots \int p(x_1)\, p(x_2) \cdots p(x_N)\, e^{-iks}\, dx_1\, dx_2 \ldots dx_N$

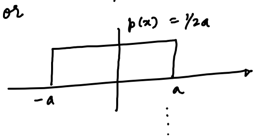
Always .... $p(s) = \dfrac{1}{(2\pi \sigma_s^2)^{1/2}} \exp\left(-\dfrac{(s-\langle s\rangle)^2}{2\sigma_s^2}\right)$

$$\tilde{\phi}(k) = \int \cdots \int p(x_1) \cdots p(x_N)\, e^{-ik(x_1+x_2+\cdots x_N)/N}\, dx_1 \ldots dx_N$$

$$= \int_{x_{1min}}^{x_{1max}} e^{-ikx_1/N} p(x_1)\, dx_1 \int_{x_{2min}}^{x_{2max}} e^{-ikx_2/N} p(x_2)\, dx_2 \cdots \int_{x_{3min}}^{x_{3max}} e^{-ikx_N/N} p(x_N)\, dx_N$$



or



$$= \left[\int_{x_{min}}^{x_{max}} e^{-ikx/N} p(x)\, dx\right]^N \quad \text{.... because } x_i \text{ is dummy (integration variable)}$$

$$= \left[\tilde{p}(k/N)\right]^N$$

$$= \left[\sum_{j=0}^{\infty} \left(-ik/N\right)^j \frac{\langle x^j\rangle}{j!}\right]^N = \left[1 + \left(\frac{-ik}{N}\right)\langle x\rangle + \left(\frac{-ik}{N}\right)^2 \frac{\langle x^2\rangle}{2} + \cancel{O\left(\frac{1}{N^3}\right)}^{\,0}\right]^N$$

Recall that $N \gg 1$

$$\tilde{\phi}(k) = \Big[1 + \underbrace{\left(\frac{-ik}{N}\right)\langle x\rangle - \frac{k^2}{2N^2}\langle x^2\rangle}_{x}\Big]^N = \left[1 + x\right]^N \underset{N \gg 1}{\simeq} e^{Nx}$$

$$\simeq e^{-ik\langle x\rangle - \frac{k^2}{2N}\langle x^2\rangle}$$

You know that the Fourier transform of Gaussian ⟶ Gaussian

$$p(s) = \frac{1}{(2\pi\sigma_s^2)^{1/2}}\, e^{-(s-\langle s\rangle)^2/2\sigma_s^2}$$

$$\phi(s) = \frac{1}{2\pi} \int_{k=-\infty}^{+\infty} e^{iks}\, \tilde{\phi}(k)\, dk$$

$$= \frac{1}{2\pi} \int_{k=-\infty}^{+\infty} e^{iks}\, e^{-ik\langle x\rangle - \frac{k^2}{2N}\langle x^2\rangle}\, dk$$

$$= \frac{1}{2\pi} \int_{k=-\infty}^{+\infty} e^{-ik(\langle x\rangle - s) - \frac{k^2}{2N}\langle x^2\rangle}\, dk$$

$$= \frac{1}{2\pi} \int_{k=-\infty}^{+\infty} e^{-\frac{\langle x^2\rangle}{2N}\left[k + \frac{(\langle x\rangle - s)iN}{\langle x^2\rangle}\right]^2} \cdot e^{-\frac{(\langle x\rangle - s)^2 N}{\langle x^2\rangle 2}}\, dk$$

Substitute $k + \underbrace{\frac{(\langle x\rangle - s)iN}{\langle x^2\rangle}}_{\text{const.}} = u$

$dk = du$

$$= \frac{1}{2\pi} \int_{u=-\infty}^{+\infty} e^{-\frac{\langle x^2\rangle}{2N}\cdot u} \cdot \underbrace{e^{-(\langle x\rangle - s)^2 N / 2\langle x^2\rangle}}_{\text{Const.}}\, du$$

$$= e^{-(s-\langle x\rangle)^2/(2\langle x^2\rangle/N)} \cdot \frac{1}{2\pi} \int_{u=-\infty}^{+\infty} e^{-\langle x^2\rangle u/2N}\, du$$

$$= e^{-(s-\langle x\rangle)^2/(2\langle x^2\rangle/N)} \cdot \frac{1}{2\pi} \cdot \sqrt{\frac{\pi\, 2N}{\langle x^2\rangle}}$$

$$= \frac{1}{\sqrt{2\pi\,\langle x^2\rangle/N}} \cdot e^{-(s-\langle x\rangle)^2/(2\langle x^2\rangle/N)}$$

(Recalling $\langle S\rangle = \sum_i \langle x_i\rangle/N = \langle x\rangle$

$$\boxed{\phi(s) = \frac{1}{\sqrt{2\pi\,(\langle x^2\rangle/N)}} \cdot e^{-(s-\langle s\rangle)^2/(2\langle x^2\rangle/N)}}$$

.... Normally distributed 's'.

Comparing this with standard Gaussian distribution..

$\sigma_s^2 = \langle x^2\rangle/N$

$\langle s\rangle = \sum_i \langle x_i\rangle/N = \langle x\rangle$

This proves CLT.