Worked examples:

(i) Take x to be distributed as Gaussian

$$x \in [-\infty, +\infty]$$

$$p_X(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-(x-\lambda)^2/2\sigma^2}$$

Mean : $\langle x \rangle_c = \lambda$

Variance : $\langle x^2 \rangle_c = \sigma^2$

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$x \in [-\infty, +\infty]$

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Mean : $\langle x \rangle_c = \lambda$

Variance : $\langle x^2 \rangle_c = \sigma^2$

Compute $\langle x^3 \rangle = \int_{x=-\infty}^{+\infty} x^3 \, p_X(x) \, dx = \int_{x=-\infty}^{+\infty} x^3 \cdot \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-(x-\lambda)^2/2\sigma^2} dx$

put $x - \lambda = u$

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Simpler alternati...

$\downarrow$

Recall $\tilde{p}(k) = e^{-ik\lambda - k^2\sigma^2/2}$

$= e^{-ik\lambda + (-ik)^2\sigma^2/2}$

$$\langle x^3 \rangle = \frac{\partial^3}{\partial(-ik)^3} \tilde{p}(k) \Big|_{k=0}$$

$\vdots$

Simplest

$\downarrow$

⋮

Simplest

$\downarrow$

$\langle x^3 \rangle =$  $+ \; 3$  .

$\langle x \rangle_c^3 \qquad 3 \langle x^2 \rangle_c \langle x \rangle$

$\lambda^3 + 3\sigma^2 \lambda$

fastest method.

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Proof of diagrams:

$$\tilde{p}(k) = \sum_{n=0}^{\infty} (-ik)^n \frac{\langle x^n \rangle}{n!}$$ Moment generator

$$\ln \tilde{p}(k) = \sum_{n=1}^{\infty} (-ik)^n \frac{\langle x^n \rangle_c}{n!}$$ "Cumulant generator"

$$\tilde{p}(k) = e^{\sum_{n=1}^{\infty} (-ik)^n \langle x^n \rangle_c / n!} \quad (1)$$

$$\tilde{p}(k) = \sum_{n=0}^{\infty} (-ik)^n \frac{\langle x^n \rangle}{n!} \quad (2)$$

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$$p(k) \quad n=0 \quad n!$$

(2)

Comparing powers of k on each sides gives the diagrams for moments.

(ii) Poisson distribution:

$$\underset{\alpha T}{p}(x) = \frac{(\alpha T)^x e^{-\alpha T}}{x!}$$

x : no. of events.

α : Mean rate

T : interval of interest.

Norm: $\sum_{x=0}^{\infty} \frac{(\alpha T)^x e^{-\alpha T}}{x!} = 1.$

$$\langle x \rangle = \sum_{x=0}^{\infty} x \frac{(\alpha T)^x e^{-\alpha T}}{x!}$$

$$= \alpha T \sum_{x=0}^{\infty} x \frac{(\alpha T)^{x-1} e^{-\alpha T}}{x!}$$

$$= \alpha T \sum_{x=0}^{\infty} \frac{1}{x!} \left[\frac{d}{d(\alpha T)} \left\{ (\alpha T)^x e^{-\alpha T} \right\} + e^{-\alpha T} (\alpha T)^x \right]$$

$$\sum_{x=0} \quad \frac{\quad}{x!}$$

$$= \alpha T \sum_{x=0}^{\infty} x \frac{(\alpha T)^{x-1} e^{-\alpha T}}{x!}$$

$$= \alpha T \sum_{x=0}^{\infty} \frac{1}{x!} \left[\frac{d}{d(\alpha T)} \left\{ (\alpha T)^{x} e^{-\alpha T} \right\} + e^{-\alpha T} (\alpha T)^{x} \right]$$

$$= \alpha T \frac{d}{d(\alpha T)} \left(\sum_{x=0}^{\infty} \frac{(\alpha T)^{x}}{x!} e^{-\alpha T} \right) + \alpha T$$

$$= \alpha T \sum_{x=0} \frac{1}{x!} \left[\frac{d}{d(\alpha T)} \left\{ (\alpha T) \, e \right\} + e \, (\alpha T) \right]$$

$$= \alpha T \, \underbrace{\cancel{\frac{d}{d(\alpha T)} \left(\sum_{x=0}^{\infty} \frac{(\alpha T)^x}{x!} e^{-\alpha T} \right)}}_{0} + \alpha T$$

$$= \alpha T$$

$$\langle x^2 \rangle = \sum_{x=0}^{\infty} x^2 \frac{(\alpha T)^x e^{-\alpha T}}{x!}$$

$$\langle x^2 \rangle = \sum_{x=0}^{\infty} x^2 \frac{(\alpha T)^x e^{-\alpha T}}{x!}$$

$$= \sum_{x=0}^{\infty} \left[\frac{x(x-1)\,(\alpha T)^x e^{-\alpha T}}{x!} + \frac{x\,(\alpha T)^x e^{-\alpha T}}{x!} \right]$$

$$= (\alpha T)^2 \sum_{x=0}^{\infty} \frac{x(x-1)\,(\alpha T)^{x-2} e^{-\alpha T}}{x!} + \sum_{x=0}^{\infty} \frac{x\,(\alpha T)^x e^{-\alpha T}}{x!}$$

$$x=0 \qquad x! \qquad\qquad x=0 \qquad x!$$

$$\Downarrow$$

$$\langle x \rangle = \alpha T$$

$$= (\alpha T)^2 \sum_{x=0}^{\infty} \left[\frac{d^2}{d(\alpha T)^2} \left(\frac{(\alpha T)^x e^{-\alpha T}}{x!} \right) + (\ S\ ) \right] + \alpha T$$

to be computed

$$= (\alpha T)^2 + \alpha T.$$

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Fastest: Characteristic function: $\tilde{p}(k) = e$

Cumulant generator: $\ln \tilde{p}(k) = \alpha T \left(e^{-ik} - 1\right)$

$$\langle x^n \rangle_c = \frac{\partial^n}{\partial(-ik)^n} \ln \tilde{p}(k) \Big|_{k=0} = \alpha T$$

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$$\langle x^n \rangle_c = \frac{\partial^n}{\partial(-ik)^n} \ln \tilde{p}(k) \Big|_{k=0} = \alpha T$$

$$\langle x^2 \rangle = \bullet \quad \bullet \quad + \quad \bullet \; \bullet$$

$$\langle x \rangle_c^2 \quad + \quad \langle x^2 \rangle_c$$

$$= (\alpha T)^2 + \alpha T.$$

$$= (\alpha T) + \alpha T.$$

(iii) Knowing how $\langle x^n \rangle$ behave, Can you compute the PDF.

$$p_X(x) = \ ?$$

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$\langle x^n \rangle = n!$ (say)

Characteristic function: $\tilde{p}(k) = \langle e^{-ikx} \rangle_x = \sum_{n=0}^{\infty} \frac{(-ik)^n \langle x^n \rangle}{n!} = \sum_{n=0}^{\infty} (-ik)^n$

... "Geometric series"

Common ratio $-ik$.

$\tilde{p}(k)$ has a closed form

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$\tilde{p}(k)$ has a closed form ...

$$\tilde{p}(k) = \frac{1}{1+ik} \quad , \quad |ik| < 1 \Rightarrow |k| < 1$$

$\tilde{p}(k)$ is also unique.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \begin{matrix} x < 1 \\ x > 1 \end{matrix}$$

$$p(x) = \frac{1}{2\pi} \int_{k=-\infty}^{+\infty} \frac{e^{ikx}}{1+ik} \, d'$$

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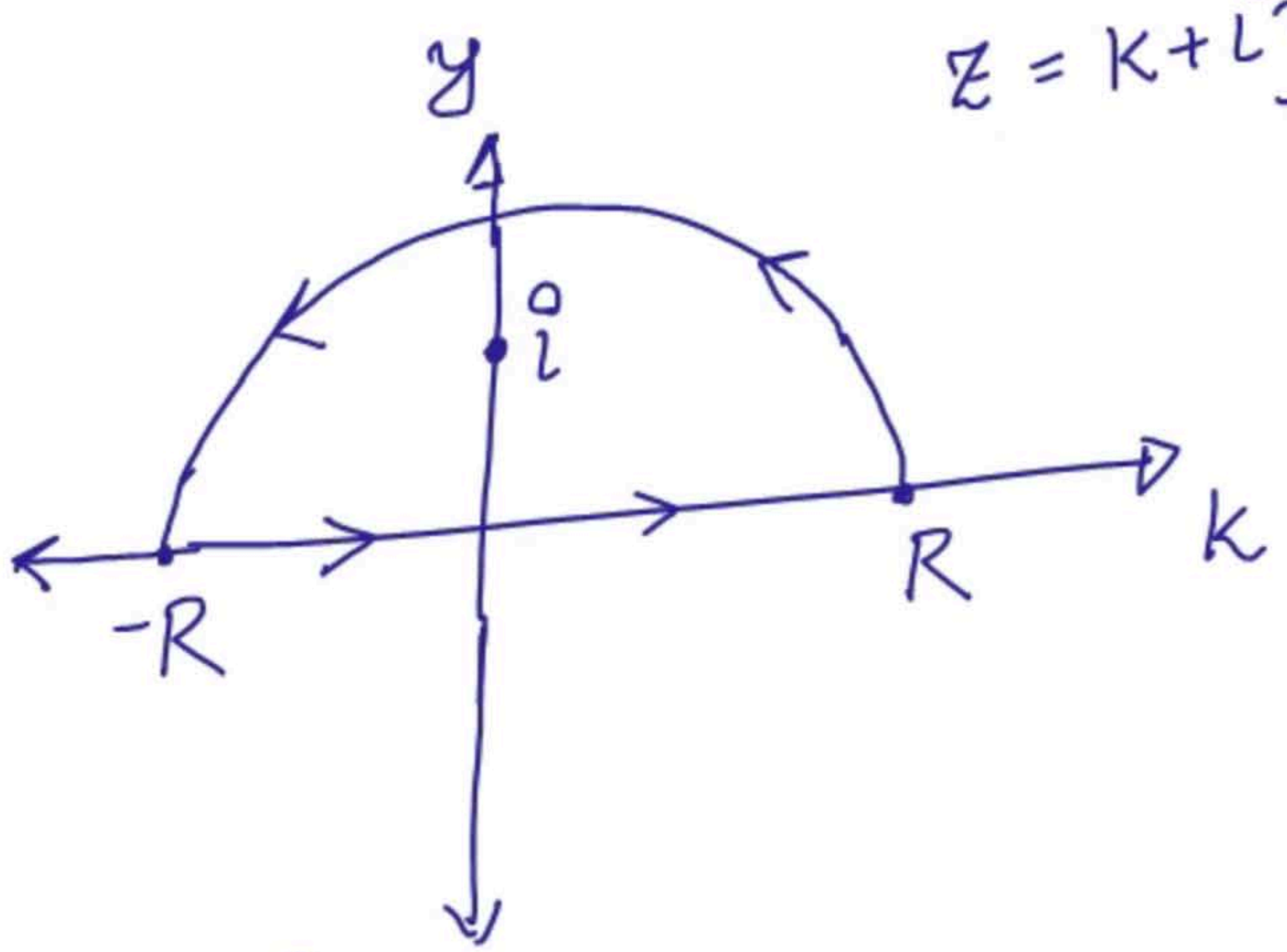
$$p(x) = \frac{1}{2\pi i}\int_{k=-\infty}^{+\infty} \frac{e^{ikx}}{(k-i)}\,dk \qquad (1)$$

z-plane

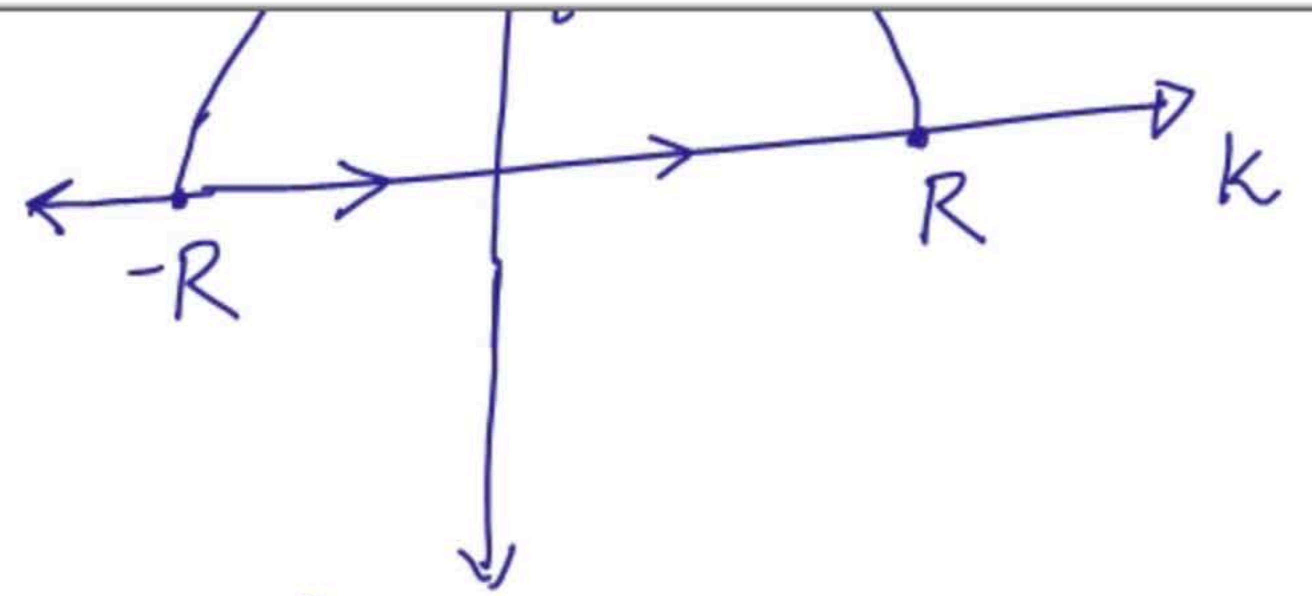
$z = k + iy$

Construct contour integral: $$I = \frac{1}{2\pi i}\oint_C \frac{e^{izx}}{(z-i)}\,dz$$



$z = i$ is a simple pole.

Construct contour integral: $I = \dfrac{1}{2\pi i} \oint_C \dfrac{e^{izx}}{(z-i)}\, dz$



$z = i$ is a simple pole.

$$I = \text{Res}\,[z=i]$$

$$= e^{i\,i\,x}$$

$$= e^{-x} \qquad \text{——— (2)}$$

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$z = i$ is a simple pole.

$$I = \text{Res}[z=i]$$

$$= e^{i i x}$$

$$\frac{1}{2\pi i}\oint_C \frac{e^{izx}}{(z-i)}\,dz = e^{-x} \qquad (2)$$

$$= \frac{1}{2\pi i}\int_{k=-R}^{+R} \frac{e^{ikx}}{(k-i)}\,dk + \frac{1}{2\pi i}\int_{\theta=0}^{\pi} \frac{e^{iRe^{i\theta}x}}{(Re^{i\theta}-i)}\, iRe^{i\theta}\,d\theta$$

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$$= e^{i\,i\,x}$$

$$\frac{1}{2\pi i}\oint_C \frac{e^{izx}}{(z-i)}\,dz = e^{-x} \quad \text{——} \quad (2)$$

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Taking $R \longrightarrow \infty$ $\quad e^{izx} = e^{iRe^{i\theta}x} = e^{iR\cos\theta x} \cdot e^{-R\sin\theta x} \longrightarrow 0$

$$-\frac{1}{2\pi i}\int_{k=-R} \frac{e}{(k-i)} \qquad \frac{}{2\pi i}\int_{\theta=0} \frac{}{(Re^{i\theta}-i)}$$

Taking $R \longrightarrow \infty$ $\qquad e^{izx} = e^{iRe^{i\theta}x} = e^{iR\cos\theta x} \cdot e^{-R\sin\theta x} \longrightarrow 0$

$$e^{-x} = \underbrace{\frac{1}{2\pi i}\int_{k=-\infty}^{+\infty} \frac{e^{ikx}}{(k-i)}\, dk}_{\downarrow \; p_X(x)} + 0$$

$z = i$ is a simple pole.

$$I = \text{Res}\,[z = i]$$

$$= e^{i i x}$$

$$\frac{1}{2\pi i}\oint_C \frac{e^{izx}}{(z-i)}\,dz = e^{-x}$$

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$$= \frac{1}{2\pi i}\int_{k=-R}^{+R} \frac{e^{ikx}}{(k-i)}\,dk + \frac{1}{2\pi i}\int_{\theta=0}^{\pi} \frac{e^{iRe^{i\theta}x}}{(Re^{i\theta}-i)}\, iRe^{i\theta}\,d\theta$$

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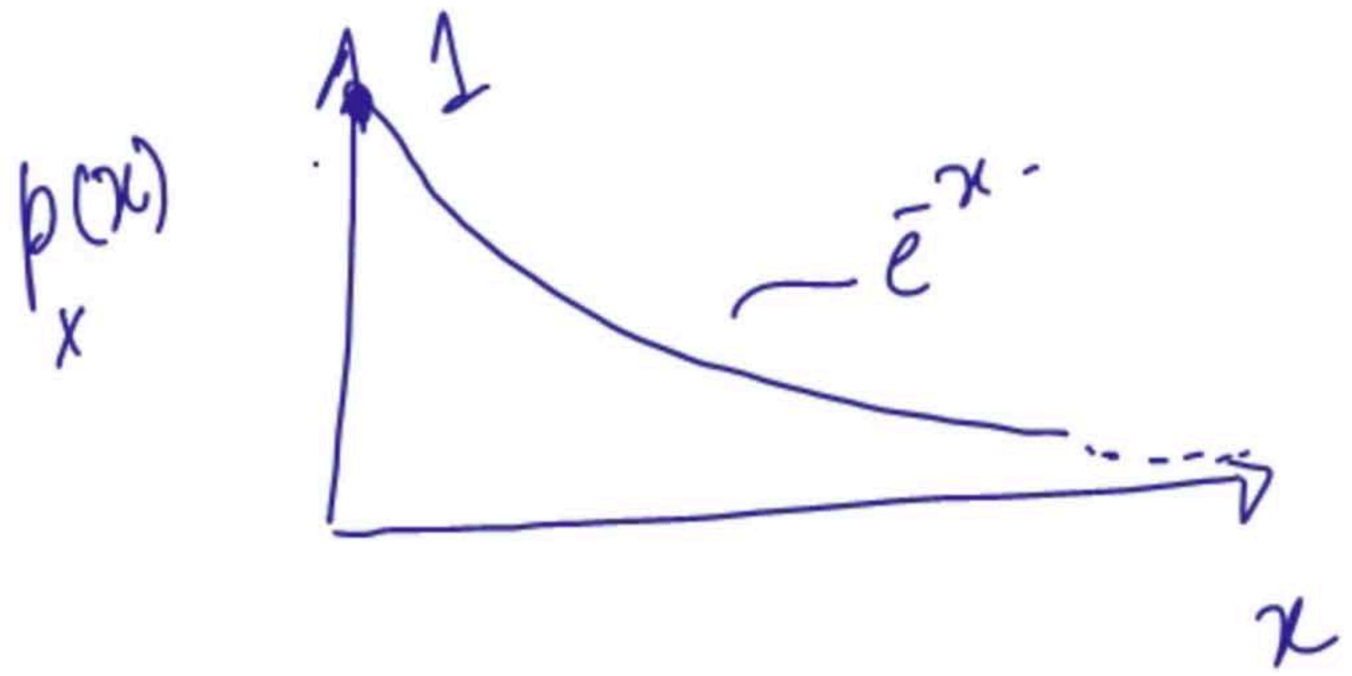
$\downarrow$

$p_X(x)$

$$p_X(x) = e^{-x} \quad [x > 0]$$



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