Many random variables.

$S_{\vec{x}} = \{-\infty < \vec{x} < +\infty\}$

$\vec{x} = (x_1, x_2, \dots x_N)$ N components of $\vec{x}$

Eg. $\vec{x} = (x, y, z)$ could be position of gas particle.

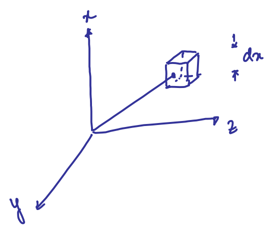
$\vec{v} = (v_x, v_y, v_z)$ ... velocity "

$p(\vec{x}, \vec{v}) \equiv$ Probability density for $\vec{x}, \vec{v}$ of a gas particle.

$p(\vec{x}, \vec{v})\, d^3\vec{x}\, d^3\vec{v} \equiv$ Probability of gas particle in the neighborhood of $[x, x+dx], [y, y+dx], [z, z+dz]$ $\Rightarrow$ positions.

$\underbrace{[v_x, v_x+dv_x], [v_y, v_y+dv_y], [v_z, v_z+dv_z]}$ $\Rightarrow$ velocities.

Volume element in 6 dimensions.



Unconditional Probability Density: $p(\vec{x})$ "Dont care about velocity"

$$\boxed{p(\vec{x}) = \int p(\vec{x}, \vec{v})\, d^3\vec{v}}$$

You don't care $\vec{v}$

Joint probability density $p(\vec{x}, \vec{v}) \sim \frac{1}{L^3} \cdot \frac{1}{V^3}$

$$\int\int p(\vec{x}, \vec{v})\, d\vec{x}^3\, d\vec{v}^3 = 1.$$

Unconditional probability density $\boxed{p(\vec{v}) = \int p(\vec{x}, \vec{v})\, d^3\vec{x}}$

Conditional probability density: $p(\vec{x}|\vec{v}) \propto p(\vec{x}, \vec{v})$

$$= \frac{p(\vec{x}, \vec{v})}{\mathcal{N}}$$

$\mathcal{N}$ can be determined from normalizing $p(\vec{x}|\vec{v})$

$$\mathcal{N} = \int p(\vec{x},\vec{v})\, d^3\vec{x}$$

$$\boxed{\begin{aligned} p(\vec{x}\,|\,\vec{v}) &= \frac{p(\vec{x},\vec{v})}{\int p(\vec{x},\vec{v})\, d^3\vec{x}} \\ p(\vec{v}\,|\,\vec{x}) &= \frac{p(\vec{x},\vec{v})}{\int p(\vec{x},\vec{v})\, d^3\vec{v}} \end{aligned}}$$

'lly

"Bayes' theorem"

$$p(x_1, x_2 \dots x_m \,|\, \underbrace{x_{m+1}, x_{m+2}, \dots x_N}_{\text{fixed}}) = \frac{p(x_1, x_2, \dots x_N)}{\int p(x_1, x_2, x_3 \dots x_N)\, dx_1 dx_2 \dots dx_m}$$

Simple consequence: x_1 & x_2 are uncorrelated.

Unconditional Probabilities = Conditional Probabilities.

Proof:

$$\underbrace{p(x_1|x_2)}_{\text{Conditional Probability density}} = \frac{p(x_1, x_2)}{\int p(x_1,x_2)\, dx_1}$$

$$= \frac{p(x_1)\, p(x_2)}{\int p(x_1)\, p(x_2)\, dx_1}$$

$$= \frac{p(x_1)\, p(x_2)}{p(x_2)\cdot 1} \qquad \because \underbrace{\int p(x_1)\, dx_1 = 1}_{\text{Normalized}}$$

$$= p(x_1)$$

= Unconditional probability density

Moments & Cumulants: For eg: $\langle x\, v_x \rangle = ?$

Characteristic function of $p(x_1, x_2 \dots x_N)$:

$x_1 = x$
$x_2 = v_x$
$\vdots$
$x_N = 0$

$$\tilde{p}(k_1, k_2 \dots k_N) = \int \dots \int dx_1 \dots dx_N\, e^{-ik_1 x_1} \dots e^{-ik_N x_N}\, p(x_1, x_2, \dots x_N)$$

N

$$= \int \cdots \int dx_1\, dx_2 \cdots dx_N \; \underline{e^{-i\sum_{j=1} k_j x_j}} \; p(x_1, x_2 \cdots x_N)$$

$$= \left\langle e^{-i\sum_{j=1}^{N} k_j x_j} \right\rangle$$

$$\left\langle x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N} \right\rangle = \frac{\partial^{n_1}}{\partial(-ik)^{n_1}} \frac{\partial^{n_2}}{\partial(-ik)^{n_2}} \cdots \frac{\partial^{n_N}}{\partial(-ik)^{n_N}} \tilde{p}(k_1, k_2 \cdots k_N) \Big|_{k_1 = k_2 = \cdots k_N = 0}$$

$$\left\langle x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N} \right\rangle_c = \frac{\partial^{n_1}}{\partial(-ik)^{n_1}} \frac{\partial^{n_2}}{\partial(-ik)^{n_2}} \cdots \frac{\partial^{n_N}}{\partial(-ik)^{n_N}} \ln \tilde{p}(k_1, k_2 \cdots k_N) \Big|_{k_1 = k_2 = \cdots k_N = 0}$$

A much simple procedure exists!

$\bullet \rightarrow x$

$\times \rightarrow V_x$

$x_1 = x \qquad n_1 = 1$

$x_2 = V_x \qquad n_2 = 1$

$n_3 = 0$

$\vdots$

$n_N = 0$

$$\langle x V_x \rangle = \underset{\langle x\rangle_c \langle V_x\rangle_c}{\bullet \;\times} + \underset{\langle x V_x\rangle_c}{(\bullet \times)} = \langle x\rangle_c \langle V_x\rangle_c + \langle x V_x\rangle_c$$

$$\langle x^2 V_x \rangle = \underset{\langle x\rangle_c^2 \langle V_x\rangle_c}{\overset{\bullet}{\bullet}\;\times} + \underset{\langle x^2\rangle_c \langle V_x\rangle_c}{(\overset{\bullet}{\bullet})\times} + \underset{2\langle x\rangle_c \langle x V_x\rangle_c}{2\,\bullet(\bullet\times)} + \underset{\langle x^2 V_x\rangle_c}{(\overset{\bullet}{\bullet}\times)}$$

Some joint moments.