Quantum Information and Computing Lecture- 26 : Implementing QFT for 3 Qubits and More

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1 Introduction

In the last lecture we discussed the implementation of QFT for the case of one and two qubits. We saw that for one qubit case the QFT is just ordinary Hadamard transform while in the case of two qubits it is implemented by having a Hadamard transform on one qubit followed by a controlled phase rotation on the other qubit. We could directly discuss generalization to n qubits which though straightforward is clumsy. Instead, we will take the case of three qubits first which will suggest the way to generalize the procedure to n qubits by observing a pattern which emerges from discussion of two and three qubits.
Recalling what we did in the last lecture, the QFT for n=2 case is given by the expression

$$\mid \tilde{x}\rangle = \frac{1}{\sqrt{2}}\left(\mid 0\rangle + (-1)^{x_0} \mid 1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(\mid 0\rangle + (-1)^{x_1} e^{\dfrac{2\pi i x_0}{4}} \mid 1\rangle\right)$$

We can see that the expression is obtained by applying a Hadamard transform on the second qubit followed by a Hadamard transform on the first qubit along with a controlled phase rotation. The phase rotation is controlled because a rotation is there only if $x_0 = 1$. The point to note in this process is the following. We cannot change the value of the second qubit before we use its value for the purpose of controlling the operations on the first qubit. This is achieved by interchanging x_1 and x_0 and then applying the Hadamard gate. Thus execution of Fourier transform requires swapping of the order of bits before application of the Hadamard and controlled B_{jk} gates.

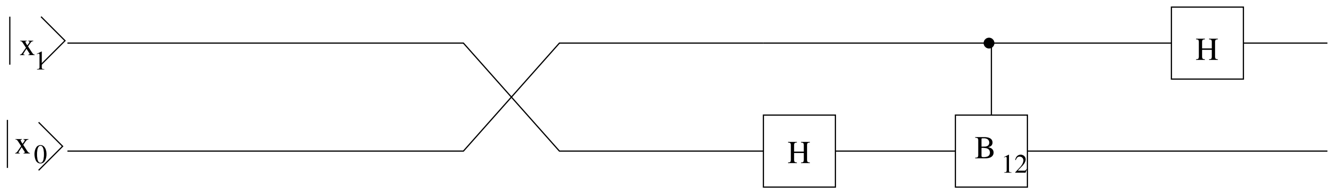

Figure 1: QFT for n=2

2 QFT for n=3

The QFT for $\mid x\rangle =\mid x_2x_1x_0\rangle$ is given by

$$\begin{aligned}\mid \tilde{x}\rangle &= \frac{1}{\sqrt{8}}\sum_{y_2,y_1,y_0=0}^{1} e^{2\pi ix(4y_2+2y_1+y_0)/8}\mid y_2y_1y_0\rangle \\ &= \frac{1}{\sqrt{2}}\sum_{y_2=0}^{1} e^{2\pi ix(4y_2/8)}\mid y_2\rangle\otimes\frac{1}{\sqrt{2}}\sum_{y_1=0}^{1} e^{2\pi ix(2y_1/8)}\mid y_1\rangle\otimes\frac{1}{\sqrt{2}}\sum_{y_0=0}^{1} e^{2\pi ix(y_0/8)}\mid y_0\rangle\end{aligned}$$

Consider the first term in the product and perform the sum over the two values that y_2 takes

$$\begin{aligned}\frac{1}{\sqrt{2}}\sum_{y_2=0}^{1} e^{2\pi ix(4y_2/8)}\mid y_2\rangle &= \frac{1}{\sqrt{2}}\left[\mid 0\rangle + e^{2\pi ix/2}\mid 1\rangle\right] \\ &= \frac{1}{\sqrt{2}}\left[\mid 0\rangle + e^{\pi i(4x_2+2x_1+x_0)}\mid 1\rangle\right]\end{aligned}$$

In the last line we have expanded x as $4x_2 + 2x_1 + x_0$. It may be observed that since x_2, x_1 and x_0 take values 0 and 1, the factor in front of the state $\mid 1\rangle$ due tp x_2 and x_1 is 1 irrespective of the value that these take. However, the multiplying factor is +1 for $x_0 = 0$ and -1 for $x_0 = 1$, i.e. the factor multiplying $\mid 1\rangle$ is $(-1)^{x_0}$. This term is simply implemented by a Hadamard transform of the first qubit. We may similarly simplify the second and the third terms in the product as follows. The second term is

$$\begin{aligned}\frac{1}{\sqrt{2}}\sum_{y_1=0}^{1} e^{2\pi ix(2y_1/8)}\mid y_2\rangle &= \frac{1}{\sqrt{2}}\left[\mid 0\rangle + e^{2\pi ix/4}\mid 1\rangle\right] \\ &= \frac{1}{\sqrt{2}}\left[\mid 0\rangle + e^{\pi i(2x_2+x_1+x_0/2)}\mid 1\rangle\right]\end{aligned}$$

Once again, the factor multiplying $\mid 1\rangle$ depends on the value of x_1 and x_0 because the factor contributed by x_2 is 1 irrespective of the value taken by x_2. The second term is thus given by

$$\frac{1}{\sqrt{2}}\left[\mid 0\rangle + (-1)^{x_1}e^{2\pi ix_0/4}\mid 1\rangle\right]$$

Looking at the above, clearly, the way to implement is to have a Hadamard transform along with a selective phase rotation by an amount $2\pi x_0/4$. The rotation is selective in the sense that the rotation is conditional upon tx_0 being equal to 1. This is implemented by a controlled B_{jk} gate with x_0 as the control. The phase of this gate is given by $2\pi/(2^{k-j+1})$. Thus the required gate for this qubit is $B_{01}^{x_0}$.
Coming to the third term in the product, one can do a similar expansion and show this term to be given by

$$\frac{1}{\sqrt{2}}\left[\mid 0\rangle + e^{2\pi i(4x_2+2x_1+x_0)/8} \mid 1\rangle\right] = \frac{1}{\sqrt{2}}\left[\mid 0\rangle + (-1)^{x_2} e^{2\pi i x_1/4} e^{2\pi i x_0/8} \mid 1\rangle\right]$$

What we require here is a Hadamard, a controlled phase rotation using x_1 as control and a phase rotation of $2\pi x_1.4$, i.e. a $B_{12}^{x_1}$ gate, followed by another controlled phase rotation $B_{12}^{x_0}$.
Putting the three terms together, the result is

$$\frac{1}{\sqrt{2}}[\mid 0\rangle + (-1)^{x_2} \mid 1\rangle] \frac{1}{\sqrt{2}}[B_{01}^{x_0}(\mid 0\rangle + (-1)^{x_1} \mid 1\rangle)] \otimes \frac{1}{\sqrt{2}}[B_{01}^{x_0} B_{12}^{x_1}(\mid 0\rangle + (-1)^{x_2} \mid 1\rangle]$$

The sequence of operation may be seen to be as follows : (i) A Hadamard gate on the *third* qubit. This is important to note as if we apply the Hadamard now, it will alter the third qubit which then cannot be used with its original value as the control. (ii) A Hadamard on the second qubit and a controlled $B_{01}^{x_0}$ and (ii) a Hadamard on the first qubit along with two controlled gates $B_{02}^{x_0} B_{12}^{x_1}$.
What it tells us is that we need to apply the gates in such a manner that the control bits for the B_{jk} gates are not changed before such gates are applied. In 3 qubit case it is achieved by interchanging the first and the third qubits.

Generalization of the above to $n-$ qubit gate is straightforward One can easily generalize the above to $n-$ qubit case. Let $\mid j\rangle = \mid j_{n-1} j_{n-2} \ldots j_0\rangle$. We have $j = j_{n-1}2^{n-1} + \ldots + j_0 2^0$ and $0.j_{n-1} + \ldots + j_0 = j_{n-1}2^{-1} + j_{n-2}2^{-2} + \ldots + j_0 2^{-n}$. Using these, we can write,

$$\mid \tilde{j}\rangle = \frac{1}{2^{n/2}}\left(\mid 0\rangle + e^{2\pi i(0.j_0)} \mid 1\rangle\right)\left(\mid 0\rangle + e^{2\pi i(0.j_1 j_0)} \mid 1\rangle\right)\ldots\otimes\left(\mid 0\rangle + e^{2\pi i(0.j_{n-1}j_{n-2})}\ldots j_0) \mid 1\rangle\right)$$

Note that each term in the above can be realized by a Hadamard transform followed by a rotation, the amount of rotation depends on the value of the other bits. Consider the m-th term on the rhs of the above product,

$$\mid 0\rangle + e^{2\pi i(0.j_m j_{m-1}\ldots j_0)} \mid 1\rangle$$

If the $m-$ th bit of j is zero, the term becomes

$$\mid 0\rangle + e^{2\pi i(0.0 j_{m-1}\ldots j_0)} \mid 1\rangle = \mid 0\rangle + e^{2\pi i(j_{m-1}\ldots j_0)/2^m} \mid 1\rangle$$

On the other hand if the $m-$ th bit is 1, this becomes

$$\mid 0\rangle - e^{2\pi i(j_{m-1}\ldots j_0)/2^m} \mid 1\rangle$$

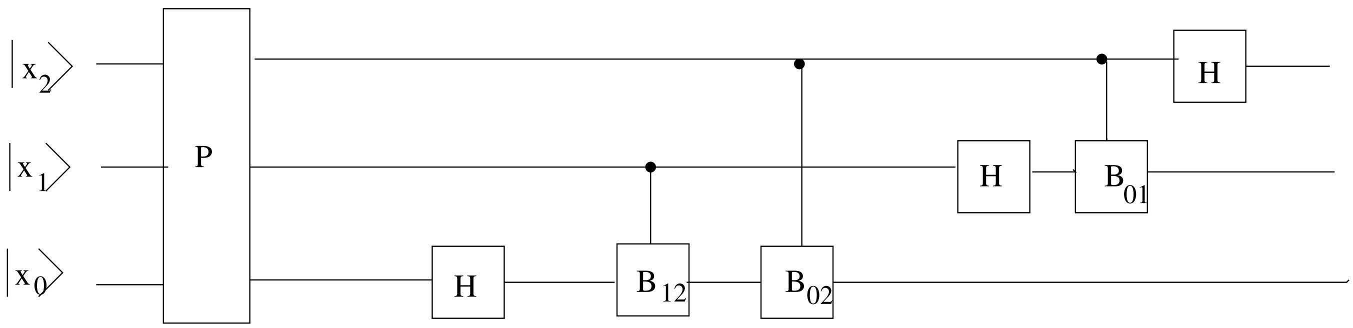

Figure 2: QFT for n=3. Here P represents a permutation which reverses the order of the lines

because $e^{2\pi i(0.j_m)} = e^{\pi i} = -1$. The amount of rotation is given by

$$2\pi(j_{m-1}\dots j_0)/2^m$$

Thus the m-th term is given by

$$\mid 0\rangle + (-1)^{j_m} e^{2\pi i(j_{m-1}\dots j_0)/2^m} \mid 1\rangle \tag{1}$$

that we had shown that in the m-th term (1) in the general expression was given by

$$\mid 0\rangle + (-1)^{j_m} \exp(2\pi i(j_{m-1}j_{m-2}\dots j_0)/2^m)$$

The phase term can be written as

$$\exp(2\pi i j_{m-1}2^{m-1}/2^m)\exp(2\pi i j_{m-2}2^{m-2}/2^m)\dots\exp(2\pi i j_0 2^0/2^m)$$

Thus a sequence of phase gates act on the state $\mid 1\rangle$ depending on the value of $j_0, j_1, \dots j_{m-1}$. The phase due to the k-th bit is $\theta = 2\pi/(2^{m-k+1})$. Thus the circuit for $n = 3$ would look as in Figure 2.