Laplace transform

Fourier integral Theorem:

Let $f_1(x)$ be an absolutely integrable function in $(-\infty, \infty)$.

$$f_{11}(x) = \begin{cases} f_1(x), & x > 0 \\ 0, & x < 0 \end{cases}, \quad f_{12}(x) = \begin{cases} 0, & x \geq 0 \\ \underline{f_1(x), \ x < 0.} & \end{cases} = \begin{cases} f_1(-x), & x > 0. \\ 0, & x \leq 0. \end{cases}$$

$$f_1(x) = f_{11}(x) + f_{12}(x).$$

Let $f_2(x) = f_1(x)$, $x \in (-\infty, \infty)$. Then $f_2(x)$ is absolutely integrable in $(-\infty, \infty)$.

$$f_2(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(t)\, e^{-i\xi t}\, dt\, e^{i\xi x}\, d\xi. \checkmark$$

$$f_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} f_1(t)\, e^{-i\xi t}\, dt\, e^{i\xi x}\, d\xi., \qquad x > 0. \checkmark$$

where $f_1(x)$ is an absolutely integrable function in $(0, \infty)$.

Let $f(x)$ be such that

$f_1(x) = e^{-cx} f(x)$, $x > 0$ with $\underline{\underline{c > 0}}$ is an absolutely integrable in $(0, \infty)$.

$\underline{\text{eg:}}$ $\underline{f(x) = e^{ax}}$ with $a < c$.

$$\Rightarrow \int_0^\infty e^{-cx} f(x)\,dx = \int_0^\infty e^{-(c-a)x}\,dx < \infty .$$

$$e^{-cx} f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \int_0^\infty e^{-ct} f(t)\, e^{-i\xi t}\,dt \cdot e^{i\xi x}\,d\xi , \quad x > 0 .$$

$$\Rightarrow \quad f(x) = \frac{e^{cx}}{2\pi}\int_{-\infty}^{\infty}\left(\int_0^\infty e^{-ct} f(t)\, e^{-i\xi t}\,dt\right) e^{i\xi x}\,d\xi , \quad x > 0 .$$

$$f(x) = \frac{e^{cx}}{2\pi}\int_{-\infty}^{\infty} \underline{\int_0^\infty f(t)\, e^{-t(c+i\xi)}\,dt} \; e^{i\xi x}\,d\xi , \quad x > 0$$

Let $c + i\xi = s$ / then

$i\,d\xi = ds$

$$f(x) = \frac{\cancel{e^{cx}}}{2\pi} \int_{c-i\infty}^{c+i\infty} \left(\int_0^\infty f(t)\, e^{-ts}\, dt \right) e^{\cancel{i}x\left(\frac{s-\cancel{c}}{\cancel{i}}\right)} \frac{ds}{i}$$

modified Fourier integral theorem

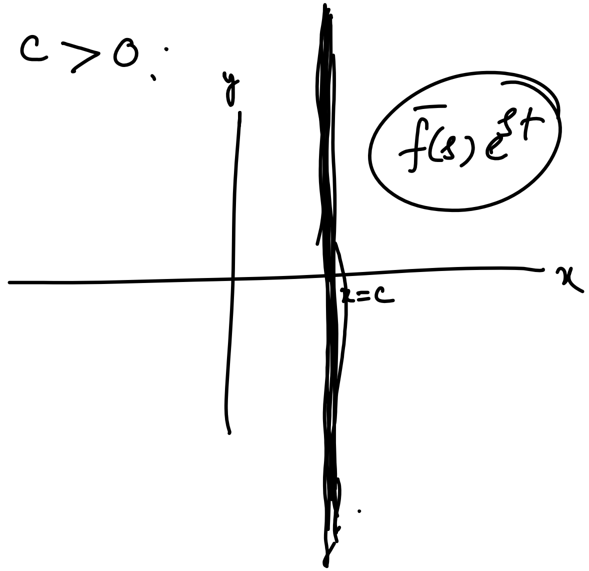
$$\boxed{\underline{f(x)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\int_0^\infty \underline{f(t)}\, e^{-ts}\, dt \right) e^{xs}\, ds \,; \qquad x > 0}$$

(Laplace transform)

<u>Defn</u>: $\mathcal{L}(f(t))(s) = \bar{f}(s) := \int_0^\infty f(t)\, e^{-ts}\, dt, \quad Re(s) > 0.$

(Inverse Laplace transform)

$$\mathcal{L}^{-1}\left(\bar{f}(s)\right) = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(s)\, e^{st}\, ds\,; \quad c>0.$$



<u>Defn:</u> (functions of exponential order)

If $f(x)$, $x>0$ such that

$$\left|f(x)\, e^{-ax}\right| \leq \underline{K} \quad \text{as } x \to \infty \text{ for } K>0, \text{ then}$$

$f(x)$ is called a function of exponential order 'a'.

ie $\quad f(x) \sim O(e^{ax}) \Leftrightarrow \lim\limits_{x\to\infty} f(x)\, e^{-ax} = \text{constant}$

If $e^{-cx} f(x)$ is absolutely integrable; then

$$\left| e^{-cx} f(x) \right| \longrightarrow 0, \text{ as } x \to \infty .$$

$$\Rightarrow \quad f(x) \sim O\left(e^{ax}\right), \text{ where } a < c .$$

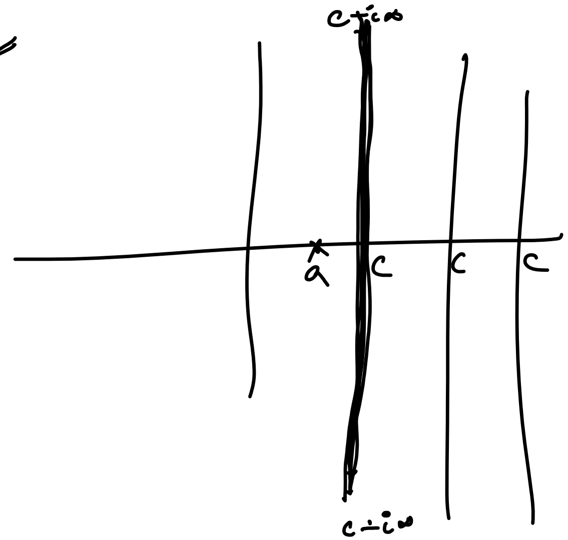
$$\lim_{x \to \infty} f(x) e^{-cx} = K > 0 .$$

$$f(x) = K e^{ax}; \quad a < c .$$

$$\lim_{x \to \infty} \left| f(x) \cdot e^{-ax} \right| = K > 0 \text{ as } x \to \infty$$

Defn: If $f(x)$ is an exponential function of order 'a', then

$$\bar{f}(s) := \int_0^\infty f(t)\, e^{-st}\, dt, \qquad \mathrm{Re}(s) > a > 0.$$



$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{f}(s)\, e^{st}\, ds, \qquad \mathrm{Re}(s) = c > a.$$

Uniqueness of Inverse Laplace transform:

$$\bar{f}(s) \to f_1(t),\ f_2(t) \Rightarrow f_1(t) = f_2(t).$$

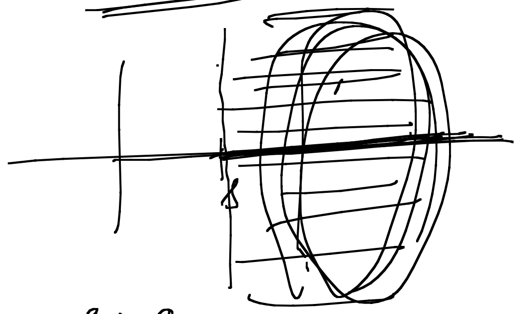
$$\mathcal{L}(0)(s) = 0$$

$$\bar{f}_1(s) = \bar{f}_2(s) \Rightarrow \int_0^\infty \left(f_1(t) - f_2(t)\right) e^{-st}\, dt = 0.$$

$$0 = \mathcal{L}^{-1}(0) = f_1(t) - f_2(t) \Rightarrow \underline{f_1(t) = f_2(t)}$$

Existence of Laplace transform:

* If $f(t)$ is continuous function in every finite interval $(0, T)$, $T > 0$ and is an exponential function of order 'a', then $\bar{f}(s)$ exists, $\mathrm{Re}(s) > a$. $f(t) = Ke^{at}$ as $t \to \infty$



Proof:

$$|\bar{f}(s)| = \left| \int_0^\infty f(t)\, e^{-st}\, dt \right| \leq \int_0^\infty e^{-st} |f(t)|\, dt$$

$$= \int_0^{T_0} e^{-st} |f(t)|\, dt + K \int_{T_0}^\infty e^{-t(s-a)}\, dt < \infty; \quad s > a.$$

$$= -\frac{M\left(e^{-sT_0} - 1\right)}{s} + K \cdot \frac{1}{e^{-T_0(s-a)}} \to 0 \text{ as } s \to \infty$$

Example s:

1. $f(t) = 1, \quad t > 0,$ then

$$\bar{f}(s) = \int_0^\infty 1 \cdot e^{-st}\, dt = -\frac{e^{-st}}{s}\Big|_{t=0}^{\infty} = \frac{1}{s}, \qquad Re(s) > 0.$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1.$$

2. $f(t) = e^{at}, \quad t > 0; \quad a \in (-\infty, \infty).$

$$\bar{f}(s) = \int_0^\infty e^{at} e^{-st}\, dt = \frac{1}{s-a}, \qquad Re(s) > a.$$

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}.$$

3. $f(t) = t^{\alpha}$; $\alpha \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$

$$\bar{f}(s) = \int_0^\infty t^{\alpha} \cdot e^{-st}\, dt$$

Let $st = x$

$s\, dt = dx$.

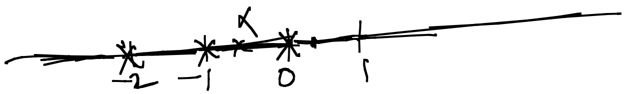
$$\bar{f}(s) = \int_0^\infty \left(\frac{x}{s}\right)^{\alpha} e^{-x}\, \frac{dx}{s}.$$

$$= \frac{1}{s^{\alpha+1}} \int_0^\infty x^{(\alpha+1)-1} e^{-x}\, dx$$

$$= \frac{1}{s^{\alpha+1}} \Gamma(\alpha+1), \quad \mathrm{Re}(s) > 0.$$

$$\Gamma(\alpha) := \int_0^\infty e^{-x} x^{\alpha-1}\, dx.$$

$$\Gamma(\alpha+1) = \int_0^\infty e^{-x} x^{\alpha}\, dx$$



$$= -\cancel{e^{-x} x^{\alpha}\Big|_0^\infty} + \alpha \int_0^\infty e^{-x} x^{\alpha-1}\, dx$$

$\alpha > 0$

$$\Gamma(\alpha+1) = \alpha\, \Gamma(\alpha) \Rightarrow \Gamma(\alpha) := \frac{1}{\alpha} \cdot \Gamma(\alpha+1)$$

If $\alpha \doteq n$, $\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1)$.

$$= n \cdot (n-1) \cdots 1 = n!$$

If $\alpha = n$, $\quad \bar{f}(s) = \dfrac{n!}{s^{n+1}}$, $\quad Re(s) > 0$.

$\lim_{x \to \infty} x^{\alpha} e^{-ax} = 0 \;\forall\, \alpha$. $\quad \underline{a > 0}$

$\Rightarrow$ x^{α} is exponential function of order b, $\quad b < a$

$\Rightarrow$ x^{α} is exponential function of order '0'.

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^{n}.$$

$$\mathcal{L}^{-1}\left(\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\right) = t^{\alpha}, \quad \alpha \in \mathbb{R} \setminus \{0,-1,-2,-3,\dots\}.$$

4. $f(t) = \sin at$; $\quad a$ is real number.

$$\bar{f}(s) = \int_0^{\infty} \sin at \cdot e^{-st}\, dt = \int_0^{\infty} \frac{e^{iat} - e^{-iat}}{2i}\, e^{-st}\, dt$$

$$= \frac{1}{2i}\left[\int_0^{\infty} e^{-t(s-ia)}\, dt - \int_0^{\infty} e^{-t(s+ia)}\, dt\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right]$$

$$= \frac{\cancel{2i}\, a}{\cancel{2i}\; s^2+a^2} = \frac{a}{s^2+a^2}, \quad Re(s) > 0.$$

5. $f(t) = \cos at, \quad a \in \mathbb{R}.$

$$\bar{f}(s) = \int_0^\infty e^{-st} \cos at \, dt = \frac{s}{s^2+a^2}, \quad Re(s) > 0$$

6. $f(t) = \sinh at,$

$$\bar{f}(s) = \int_0^\infty \sinh at \, e^{-st} \, dt = \int_0^\infty \frac{e^{at} - e^{-at}}{2} e^{-st} \, dt$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{a}{s^2-a^2}, \quad Re(s) > a.$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at.$$

7. $f(t) = \cosh at, \quad a \in \mathbb{R}$

$$\bar{f}(s) = \frac{s}{s^2-a^2}, \quad Re(s) > a..$$

$$\mathcal{L}\left(\frac{s}{s^2-a^2}\right) = \cosh at.$$

$$\bar{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$\lim_{s \to \infty} |\bar{f}(s)| = 0$. if $f(t)$ is continuous and exponential function of same order.

If $\bar{f}(s) = s$ or s^2, then since $\lim_{s \to \infty} |\bar{f}(s)| = \infty$; $\bar{f}(s)$ is not a Laplace transform of any continuous function.

If $f(t) = e^{at^2}$, $a > 0$, then

$$\bar{f}(s) = \int_0^\infty e^{at^2} e^{-st} \, dt. = \infty.$$

$$\lim_{t \to \infty} e^{at^2} e^{-st}. = \lim_{t \to \infty} e^{at^2 - st} = \infty \quad \forall s.$$

$\Rightarrow$ $\bar{f}(s)$ does not exist for e^{at^2}.

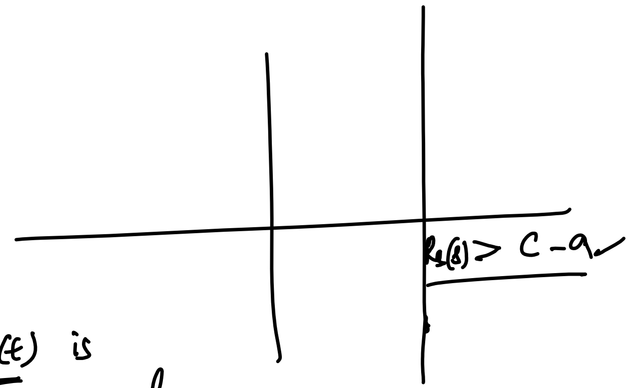
Properties of Laplace transforms:

1. If $\bar{f}(s) = \mathcal{L}(f(t))(s)$ then $\mathcal{L}(e^{-at} f(t))(s) = \bar{f}(s+a)$, $a \in \mathbb{R}$.

$$\mathcal{L}\left(e^{-at} f(t)\right)(s) = \int_0^\infty e^{-at} f(t)\, e^{-st}\, dt.$$

$$= \int_0^\infty e^{-t(s+a)} f(t)\, dt$$

$$= \bar{f}(s+a) \checkmark$$



$e^{-at} f(t)$ is exponential of order $c-a$.

eg: 1. $\mathcal{L}\left(t^n e^{-at}\right) = \dfrac{n!}{(s+a)^{n+1}}$ since $\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$

2. $\mathcal{L}\left(e^{-at}\sin bt\right) = \dfrac{b}{(s+a)^2 + b^2}$

3. $\mathcal{L}\left(e^{-at}\cos bt\right) = \dfrac{s+a}{(s+a)^2 + b^2}$.

(2.) If $\mathcal{L}(f(t))(s) = \bar{f}(s)$, then $\mathcal{L}\left(f(t-a)\,H(t-a)\right)(s) = e^{-as}\,\bar{f}(s)$.

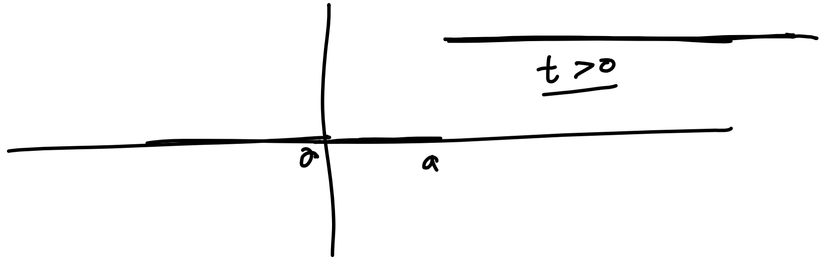
$$\mathcal{L}\left(f(t-a)\,H(t-a)\right)(s) = \int_0^\infty f(t-a)\,\underline{H(t-a)}\,e^{-st}\,dt.$$

$$= \int_a^\infty f(t-a)\,e^{-st}\,dt$$

$t-a = x$
$dt = dx$

$$= \int_0^\infty f(x)\, e^{-s(a+x)}\, dx$$

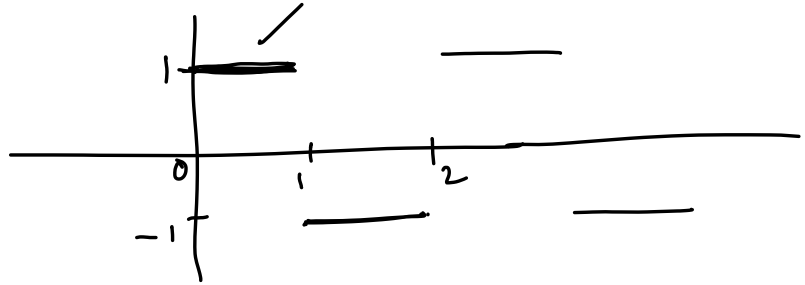
$$= e^{-sa}\, \bar{f}(s).$$



eg: 1. $f(t) = 1$, $\quad \mathcal{L}(H(t-a)) = e^{-sa}\,\frac{1}{s}$.

2. If $f(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \\ 1, & t > 2 \end{cases}$

$f(t)$



$$f(t) = 1 - 2H(t-1) + 2H(t-2)$$

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \\ 1 & t > 2 \end{cases}$$

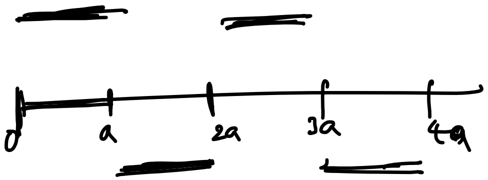
$$\mathcal{L}(f(t))(s) = \mathcal{L}\left(1 - 2H(t-1) + 2H(t-2)\right)(s)$$

$$= \mathcal{L}(1)(s) - 2\mathcal{L}(H(t-1))(s) + 2\mathcal{L}(H(t-2))(s)$$

$$\bar{f}(s) = \frac{1}{s} - 2\,\frac{e^{-s}}{s} + 2\,\frac{e^{-2s}}{s}.$$

(3) If $\mathcal{L}(f(t))(s) = \bar{f}(s)$, then $\mathcal{L}(f(at)) = \frac{1}{a}\,\bar{f}\left(\frac{s}{a}\right)$, $a \in \mathbb{R}$

$$\mathcal{L}(f(at)) = \int_0^\infty f(at)\, e^{-st}\, dt = \frac{1}{a}\int_0^\infty f(x)\, e^{-\frac{s}{a}x}\, dx \qquad at = x,\quad a\,dt = dx$$



$$= \frac{1}{a}\,\bar{f}\left(\frac{s}{a}\right), \quad \forall\, a \in \mathbb{R}.$$

Example of periodic function:

$$f(t) = H(t) - 2H(t-a) + 2H(t-2a) - 2H(t-3a) + 2H(t-4a) - \cdots$$

$$\bar{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt = \frac{1}{s} - 2\,\frac{e^{-as}}{s} + 2\,\frac{e^{-2as}}{s} - 2\,\frac{e^{-3as}}{s} + \cdots$$

$$= \frac{1}{s}\left(1 - 2e^{-as}\left(1 - e^{-as} + e^{-2as} + \cdots\right)\right)$$

$$|e^{-as}| < 1$$

$$= \frac{1}{s}\left(1 - 2e^{-as} \cdot \frac{1}{1+e^{-as}}\right)$$

$$= \frac{1}{s}\,\frac{1-e^{-as}}{1+e^{-as}} = \frac{1}{s}\,\frac{\cancel{e^{-as/2}}\left(e^{as/2} - e^{-as/2}\right)}{\cancel{e^{-as/2}}\left(e^{as/2} + e^{-as/2}\right)}$$

$$\boxed{\bar{f}(s) = \frac{1}{s}\tanh\left(\frac{sa}{2}\right).}$$

4. If $f(t)$ is a periodic function with period 'a'.

Then $\mathcal{L}(f(t))$ when it exists, is

$$\mathcal{L}(f(t)) = \frac{1}{1-e^{-as}} \cdot \int_0^a e^{-st} f(t)\, dt.$$

Sol:- If $f(t) = f(t+a)$, $\forall$ $\underline{t>0}$ $\underline{a>0}$.

$$\mathcal{L}(f(t))(s) = \bar{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt.$$

$$= \int_0^a f(t)\, e^{-st}\, dt + \int_a^\infty f(t)\, e^{-st}\, dt$$

$t-a=x$, $dt = dx$

$$= \int_0^a f(t)\, e^{-st}\, dt + \int_0^\infty f(x+a)\, e^{-s(x+a)}\, dx$$

$$\bar{f}(s) = \int_0^a f(t)\, e^{-st}\, dt + e^{-sa}\, \bar{f}(s)$$

$$\Rightarrow \boxed{\bar{f}(s) = \frac{1}{1-e^{-sa}} \int_0^a f(t)\, e^{-st}\, dt.}$$

5 If $f(t) = O(e^{at})$ as $t \to \infty$, then

$\int_0^\infty f(t)\, e^{-st}\, dt$ is uniformly convergent w.r.t s for $\underline{s > a}$.

proof: $\left| f(t)\, e^{-st} \right| \le \left| K e^{at} \cdot e^{-st} \right| \le K e^{-t(s-a)} \le K e^{-t(a_1-a)}$, if $a_1 \le s$ with $a_1 > a$.

$$\left| \int_0^\infty f(t)\, e^{-st}\, dt \right| \le \int_0^\infty \left| f(t)\, e^{-st} \right| dt \le K \int_0^\infty e^{-t(a_1-a)}\, dt < \infty, \quad \forall\, s \ge a_1 > a.$$

$$\frac{d}{dx} \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} f_n'(x)$$

(i) $$\frac{d}{ds} \bar{f}(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty \frac{d}{ds}\left(e^{-st} f(t) \right) dt. \checkmark$$

(ii) $$\int_s^\infty \bar{f}(s)\, ds = \int_s^\infty \int_0^\infty f(t)\, e^{-st}\, dt\, ds = \int_0^\infty \int_s^\infty e^{-st}\, ds\, f(t)\, dt \checkmark$$

(6) If $\bar{f}(s) = \mathcal{L}(f(t))(s)$, then

$$\mathcal{L}(f'(t))(s) = s\bar{f}(s) - f(0)$$

$$\mathcal{L}(f''(t))(s) = s^2\bar{f}(s) - sf(0) - f'(0)$$

$$\vdots \qquad \vdots$$

$$\mathcal{L}(f^{(n)}(t))(s) = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

proof:

$$\mathcal{L}(f'(t))(s) = \int_0^\infty f'(t)e^{-st}\,dt = f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt$$

$$= s\bar{f}(s) - f(0).$$

$$\mathcal{L}(f''(t))(s) = \int_0^\infty f''(t)e^{-st}\,dt = f'(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f'(t)e^{-st}\,dt$$

$$= -f'(0) + s\left(s\bar{f}(s) - f(0)\right).$$

$$= s^2\bar{f}(s) - sf(0) - f'(0).$$

By induction, we can show that

$$\mathcal{L}\left(f^{(n)}(t)\right)(s) = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

7 If $\bar{f}(s) = \mathcal{L}(f(t))(s)$ and $\bar{g}(s) = \mathcal{L}(g(t))(s)$. Then

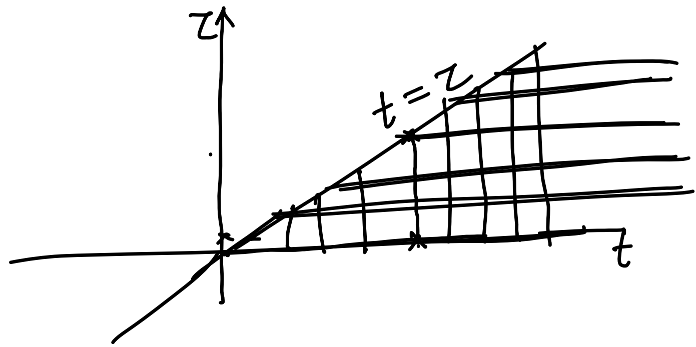
$$\mathcal{L}\left(f*g(t)\right)(s) = \mathcal{L}\left(\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right)(s) = \bar{f}(s)\cdot\bar{g}(s).$$

<u>Defn</u>: $f*g(t) := \int_0^t f(\tau)\,g(t-\tau)\,d\tau$

$f*g(t) = g*f(t)$ ✓

Proof:

$$\mathcal{L}\left(f * g(t)\right)(s) = \int_0^\infty \int_0^t f(\tau)\, g(t-\tau)\, d\tau \; e^{-st}\, dt.$$



$$= \int_0^\infty \int_\tau^\infty f(\tau)\, g(t-\tau)\, e^{-st}\, dt \; d\tau$$

$$= \int_0^\infty f(\tau) \int_\tau^\infty g(t-\tau)\, e^{-st}\, dt \; d\tau$$

Let $t - \tau = x \qquad dt = dx$

$$= \int_0^\infty f(\tau) \int_0^\infty g(x)\, e^{-s(x+\tau)}\, dx\, d\tau$$

$$= \int_0^\infty f(\tau)\, e^{-s\tau}\, d\tau \int_0^\infty g(x)\, e^{-sx}\, dx$$

$$\underline{\mathcal{L}(f*g(t))(s) = \bar{f}(s) \cdot \bar{g}(s).}$$

$$\beta(m,n) := \int_0^1 x^{m-1}(1-x)^{n-1}\, dx, \quad m>0,\ n>0$$

<u>eg</u>: show that $\beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

Let $f(t) = t^{m-1}$, $g(t) = t^{n-1}$.

$$\bar{f}(s) = \frac{\Gamma(m)}{s^m}, \quad \bar{g}(s) = \frac{\Gamma(n)}{s^n}.$$

Since $\mathcal{L}(f*g(t))(s) = \bar{f}(s) \cdot \bar{g}(s)$

$$\mathcal{L}^{-1}\left(\frac{\Gamma(m)\cdot\Gamma(n)}{s^{m+n}}\right) = \mathcal{L}^{-1}\Big(\mathcal{L}\big(f*g(t)\big)(s)\Big)$$

$$= f*g(t).$$

$$\mathcal{L}^{-1}\mathcal{L}\left(t^{m+n-1}\right) = \mathcal{L}^{-1}\left(\frac{\Gamma(m+n)}{s^{m+n}}\right)$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s^{m+n}}\right) = \frac{t^{m+n-1}}{\Gamma(m+n)} \checkmark$$

$$\Gamma(m)\,\Gamma(n)\,\mathcal{L}^{-1}\left(\frac{1}{s^{m+n}}\right) = f*g(t)$$

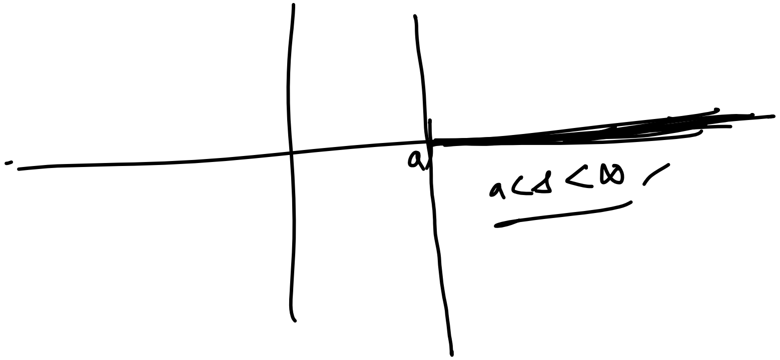
$$\Rightarrow \boxed{\int_0^1 \tau^{m-1}(t-\tau)^{n-1}\,d\tau = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\,t^{m+n-1}}, \quad \forall \quad t \geq 0$$

If $t=1$,

$$\Rightarrow \quad \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

(8) If $\bar{f}(s) = \mathcal{L}(f(t))(s)$, then

$$\mathcal{L}(t^n f(t))(s) = (-1)^n \frac{d^n}{ds^n}\left(\bar{f}(s)\right), \quad n = 0, 1, 2, 3, \dots$$



Proof: $n = 1$, $\mathcal{L}(t f(t))(s) = \int_0^\infty t\, f(t)\, e^{-st}\, dt$

$$= -\int_0^\infty f(t) \frac{d}{ds}\left(e^{-st}\right) dt$$

$$= -\frac{d}{ds}\int_0^\infty f(t)\, e^{-st}\, dt$$

$$\mathcal{L}(t f(t))(s) = -\frac{d}{ds}\left(\bar{f}(s)\right). \checkmark$$

$\underline{n=k}$, Assume that the result is true.

$$\mathcal{L}\left(t^{k+1} f(t)\right)(s) = \int_0^\infty t^{k+1} f(t)\, e^{-st}\, dt$$

$$= -\int_0^\infty t^k f(t) \frac{d}{ds}\left(e^{-st}\right) dt$$

$$= -\frac{d}{ds}\left(\mathcal{L}\left(t^k f(t)\right)(s)\right)$$

$$= -\frac{d}{ds}\left((-1)^k \frac{d^k}{ds^k} \bar{f}(s)\right)$$

$$= (-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}}\left(\bar{f}(s)\right)$$

By induction, the result is true.

<u>eg</u>: 1. $$\mathcal{L}\left(t^n e^{-at}\right) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\left(e^{-at}\right)(s)\right)$$

$$= (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s-a}\right)$$

$$= (-1)^n (-1)^n \cdot \frac{n!}{(s-a)^{n+1}}$$

$$= \frac{n!}{(s-a)^{n+1}} \checkmark$$

2. $$\mathcal{L}\left(t \underset{\substack{\text{or} \\ \sin at}}{\cos at}\right)(s) = -\frac{d}{ds}\left(\mathcal{L}\left(\underset{\substack{\text{or} \\ \sin at}}{\cos at}\right)(s)\right)$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+a^2} \text{ or } \frac{a}{s^2+a^2}\right)$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2} \quad \text{or} \quad \frac{2as}{(s^2+a^2)^2}$$

(9) If $\bar{f}(s) = \mathcal{L}(f(t))(s)$, then

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty \bar{f}(s)\, ds.$$

Proof. $$R.H.S = \int_s^\infty \underline{\int_0^\infty f(t)\, e^{-st}\, dt}\, ds = \int_0^\infty f(t) \underline{\int_s^\infty e^{-st}\, ds}\, dt$$

$$= \int_0^\infty \left. \frac{-e^{-st}}{t} \right|_{s=s}^{\infty} f(t)\, dt$$

$$= \int_0^\infty \frac{e^{-st}}{t} f(t)\, dt = \mathcal{L}\left(\frac{f(t)}{t}\right)(s) = \text{L.H.S}$$

<u>eg:</u> $$\mathcal{L}\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{s^2+a^2}\, ds = a \int_s^\infty \frac{ds}{s^2+a^2}$$

$s/a = x$

$$= \int_s^\infty \frac{d(s/a)}{1+(s/a)^2}$$

$$= \int_{s/a}^\infty \frac{dx}{1+x^2} = \tan^{-1} x \Big|_{s/a}^{\infty} = \underline{\frac{\pi}{2} - \tan^{-1}(s/a) = \tan^{-1}(a/s)}.$$

(10) If $\bar{f}(s) = \mathcal{L}(f(t))(s)$ Then

$$\mathcal{L}\left\{ \int_0^t f(\tau)\, d\tau \right\} = \frac{1}{s} \bar{f}(s).$$

proof: Let $g(t) = \int_0^t f(\tau)\, d\tau$. Then $g(0) = 0$

$$g'(t) = f(t); \quad t > 0.$$

$$\Rightarrow \mathcal{L}(g'(t))(s) = \bar{f}(s)$$

$$\Rightarrow s\,\bar{g}(s) - \cancel{g(0)} = \bar{f}(s).$$

$$\Rightarrow \bar{g}(s) = \frac{1}{s} \bar{f}(s). \checkmark$$

eg: 1. $\mathcal{L}\left(\int_0^t \tau^n e^{-a\tau}\, d\tau\right)$

Since $\mathcal{L}\left(t^n e^{-at}\right)(s) = \dfrac{n!}{(s+a)^{n+1}}$,

$$\mathcal{L}\left(\int_0^t \tau^n e^{-a\tau}\, d\tau\right) = \frac{1}{s}\,\frac{n!}{(s+a)^{n+1}}.$$

2. $\mathcal{L}\left(\int_0^t \dfrac{\sin a\tau}{\tau}\, d\tau\right).$

$$= \frac{1}{s}\,\mathcal{L}\left(\frac{\sin at}{t}\right)(s) = \frac{1}{s}\tan^{-1}(a/s).$$

Methods of finding inverse Laplace transform:

1. Partial fraction method.

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right)(t) = e^{at}$$

$$\bar{f}(s) = \frac{\bar{p}(s)}{\bar{q}(s)}, \quad \deg(\bar{p}(s)) < \deg \bar{q}(s).$$

$\bar{f}(s) =$ sum of fractions $\frac{1}{(s-s_i)^k}$, $k \in \mathbb{N}$ with $\bar{q}(s_i) = 0$.

eg: 1. $\mathcal{L}^{-1}\left\{\frac{1}{s(s-a)}\right\}(t)$

$$= -\frac{1}{a}\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s-a}\right\}(t)$$

$$= -\frac{1}{a}\left[\mathcal{L}^{-1}\left(\frac{1}{s}\right)(t) - \mathcal{L}^{-1}\left(\frac{1}{s-a}\right)(t)\right]$$

$$= -\frac{1}{a} + \frac{1}{a} e^{at}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s(s-a)}\right)(t) = \frac{(e^{at}-1)}{a} \quad ✓$$

2. $$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\}(t) = ?$$

$$- \mathcal{L}^{-1}\left\{\frac{1}{a}\frac{a}{s^2+a^2} - \frac{1}{b}\frac{b}{s^2+b^2}\right\}(t) \cdot \frac{1}{(b^2-a^2)}$$

$$= \frac{1}{(b^2-a^2)}\left\{\frac{1}{a} \cdot \sin at - \frac{1}{b}\sin bt\right\}$$

$$= \frac{1}{(b^2-a^2)}\left(\frac{\sin at}{a} - \frac{\sin bt}{b}\right) \quad ✓$$

3. $$\mathcal{L}^{-1}\left\{\frac{s+7}{s^2+2s+5}\right\}(t) = ?$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2} + \frac{6}{(s+1)^2+2^2}\right\}$$

$$= e^{-t}\cos 2t + 3\cdot e^{-t}\sin 2t$$

$$= e^{-t}(\cos 2t + 3\sin 2t).$$

4. $$\mathcal{L}^{-1}\left\{\frac{2s^2+5s+7}{(s-2)(s^2+4s+13)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-2} + \frac{s+2}{(s+2)^2+3^2} + \frac{1}{3}\frac{3}{(s+2)^2+3^2}\right\}$$

$$= e^{2t} + e^{-2t}\cos 3t + \frac{1}{3}e^{-2t}\sin 3t \checkmark$$

2. Convolution Theorem to find inversion.

$$(f * g(t))(s) = \mathcal{L}^{-1}\left(\bar{f}(s) \cdot \bar{g}(s)\right) \checkmark$$

eg: 1. $\mathcal{L}^{-1}\left\{\frac{1}{s(s-a)}\right\}(t)$

$$= \mathcal{L}^{-1}\left\{\bar{1} \cdot \overline{e^{at}}\right\}$$

$$= \int_0^t e^{a(t-\tau)} \, d\tau = e^{at} \left.\frac{e^{-a\tau}}{-a}\right|_0^t$$

$$= \frac{-1}{a} + \frac{e^{at}}{a}$$

$$= \frac{e^{at}-1}{a}. \checkmark$$

2. $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = \frac{1}{a}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{a}{s^2+a^2}\right\}$

$\mathcal{L}\{t\} = \int_0^\infty t\, e^{-st}\,dt$

$= \frac{1}{s^2}$

$\frac{a}{s^2+a^2}$

$$= \frac{1}{a}\mathcal{L}^{-1}\left\{\overline{t}\cdot\overline{\sin at}\right\}$$

$$= \frac{1}{a}\int_0^t (t-\tau)\sin a\tau\, d\tau$$

$$= \frac{t}{a}\int_0^t \sin a\tau\, d\tau - \frac{1}{a}\int_0^t \tau\sin a\tau\, d\tau.$$

$$= \frac{t}{a}\cdot -\frac{\cos a\tau}{a}\Big|_0^t - \frac{1}{a^2}\left[-\cos a\tau\cdot\tau\Big|_0^t + \int_0^t \cos a\tau\, d\tau\right]$$

$$= \cancel{-\frac{t}{a^2}\cos a\tau} + \frac{t}{a^2} \cancel{- \frac{1}{a^2}\left(-t\cos at\right)} - \frac{1}{a^2}\frac{\sin a\tau}{a}\Bigg|_{\tau=0}^{\tau=t}$$

$$= \frac{t}{a^2} - \frac{\sin at}{a^3} = \underline{\frac{1}{a^2}\left(t - \frac{\sin at}{a}\right)} \checkmark$$

$$t-\tau = x$$

Remark:

Property 10:

$$\left(\int_0^t f(\tau)\,d\tau\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\bar{f}(s)\right) = 1 * f(t) = \int_0^t f(t-\tau)\,d\tau = \int_0^t f(x)\,dx$$

3 Heaviside expansion to find inversion.

Suppose $\bar{f}(s) = \mathcal{L}(f(t))(s)$

$$f(t) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}, \quad t > 0$$

$$\mathcal{L}\left(\frac{t^n}{n!}\right) = \frac{1}{s^{n+1}}$$

$$\mathcal{L}(f(t))(s) = \sum_{n=0}^{\infty} \frac{c_n}{s^{n+1}}.$$

$$\mathcal{L}(f(t))(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$\boxed{\mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} \frac{c_n}{s^{n+1}}\right) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}}$$

3 Heavyside expansion therem: If $\bar{f}(s) = \dfrac{\bar{p}(s)}{\bar{q}(s)}$, where $\bar{p}(s)$, $\bar{q}(s)$ are polynomials

such that $n = \deg(\bar{q}(s)) > \deg(\bar{p}(s))$

Assume that $\bar{q}(s) = 0$ has distinct roots $\alpha_1, \alpha_2, \cdots \alpha_n$.

then $$\mathcal{L}^{-1}\left(\bar{f}(s)\right) = \sum_{k=1}^{n} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} e^{\alpha_k t}.$$

proof by defn of inverse Laplace transform as a contour integral

$$\bar{f}(s) = \frac{\bar{p}(s)}{\bar{q}(s)}$$ $\bar{q}(s)$ infinite distinct zeros

$$\mathcal{L}^{-1}\left(\bar{f}(s)\right) = \sum_{k=1}^{\infty} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} e^{\alpha_k t}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(s)\, e^{st}\, ds = \text{Residues at poles}$$

proof:

$$\bar{q}(s) = a_0 (s-\alpha_1)(s-\alpha_2) \cdots (s-\alpha_n).$$

$$\bar{f}(s) = \frac{\bar{p}(s)}{\bar{q}(s)} = \sum_{k=1}^{n} \frac{A_k}{(s-\alpha_k)},$$ where A_k is a constant.

$$\Rightarrow \bar{p}(s) = \sum_{k=1}^{n} \frac{A_k}{(s-\alpha_k)} \bar{q}(s) = \sum_{k=1}^{n} a_0 A_k (s-\alpha_1)(s-\alpha_2) \cdots (s-\alpha_{k-1})(s-\alpha_{k+1}) \cdots (s-\alpha_n).$$

$$\bar{p}(\alpha_k) = a_0 A_k (\alpha_k-\alpha_1)(\alpha_k-\alpha_2) \cdots (\alpha_k-\alpha_{k-1})(\alpha_k-\alpha_{k+1}) \cdots (\alpha_k-\alpha_n) \neq 0$$

$$\bar{q}'(s) = a_0 \sum_{-}^{n} (s-\alpha_1 \quad s-\alpha_2 \; --- \; (s-\alpha_{k-1})(s-\alpha_{k+1}) --- (s-\alpha_n).$$

$$\bar{q}(\alpha_k) = a_0 (\alpha_k - \alpha_1)(\alpha_k - \alpha_2) ---- (\; -\alpha_{k-1})(\alpha_k - \alpha_{k+1}) ---- (\alpha_k - \alpha_n)$$

$$A_k = \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)}, \quad k = 1, 2, 3, --, n.$$

$$\bar{f}(s) = \sum_{k=1}^{n} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} \frac{1}{(s-\alpha_k)}$$

$$f(t) = \mathcal{L}^{-1}(\bar{f}(s))(t) = \sum_{k=1}^{n} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} e^{\alpha_k t}.$$

eg: 1. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-3s+2}\right\}$

$\bar{q}'(s) = 2s-3$

$\bar{p}(s) = s, \quad \bar{q}(s) = s^2-3s+2 = (s-2)(s-1).$

$$\mathcal{L}^{-1}\left\{\frac{\bar{p}(s)}{\bar{q}(s)}\right\} = \mathcal{L}^{-1}\left\{\frac{A_1}{s-1} + \frac{A_2}{s-2}\right\}$$

$$A_1 = \frac{\bar{p}(1)}{\bar{q}'(1)} = \frac{1}{-1} = -1$$

$$A_2 = \frac{\bar{p}(2)}{\bar{q}'(2)} = \frac{2}{1} = 2$$

$$= \mathcal{L}^{-1}\left\{-\frac{1}{s-1} + \frac{2}{s-2}\right\}$$

$$= -e^{t} + 2e^{2t}$$

(2) If $\alpha = \sqrt{\frac{s}{a}}$, then

find $\mathcal{L}^{-1}\left[\frac{\cosh \alpha x}{s \cosh \alpha l}\right]$, where $x, l \in \mathbb{R}$.

$\bar{p}(s) = \underline{\cosh \sqrt{\frac{s}{a}}\, x}$, $\quad \bar{q}(s) = s \cosh \sqrt{\frac{s}{a}}\, l$.

$s_{-1} = 0$, $\quad s_k = -\frac{(2k+1)^2 \pi^2 a}{4 l^2}$

Calculate: $\frac{\bar{p}(s_k)}{\bar{q}'(s_k)}$, $\quad k = 0, 1, 2, 3, \ldots$

$\underline{\cosh \sqrt{\frac{s}{a}}\, l = 0}$. $\quad \cosh \frac{i(2k+1)\pi}{2}$

$k = 0, 1, 2, \ldots$

$\sqrt{\frac{s}{a}}\, l = \frac{i(2k+1)\pi}{2}$, $k = 0, 1, 2, \ldots$

$s_k = -\frac{(2k+1)^2 \pi^2}{4} \frac{a}{l^2}$, $k = 0, 1, 2, \ldots$

$\bar{q}'(s) = \underline{\cosh \sqrt{\frac{s}{a}}\, l + \sqrt{s} \sinh \sqrt{\frac{s}{a}}\, l \cdot \frac{l}{\sqrt{a}} \cdot \frac{1}{2}}$

$\bar{q}'(s_k)$

$$\mathcal{L}^{-1}\left\{\frac{\cosh \alpha x}{s \cosh \alpha l}\right\} = \mathcal{L}^{-1}\left\{\sum_{k=-1}^{\infty} \frac{\bar{p}(s_k)}{\bar{q}'(s_k)} \cdot \frac{1}{(s-s_k)}\right\}$$

$$= \sum_{k=-1}^{\infty} \frac{\bar{p}(s_k)}{\bar{q}'(s_k)} e^{s_k t}.$$

---

④ Laplace inversion for a general function $\bar{f}(s)$.

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(s)\, e^{st}\, ds,$$

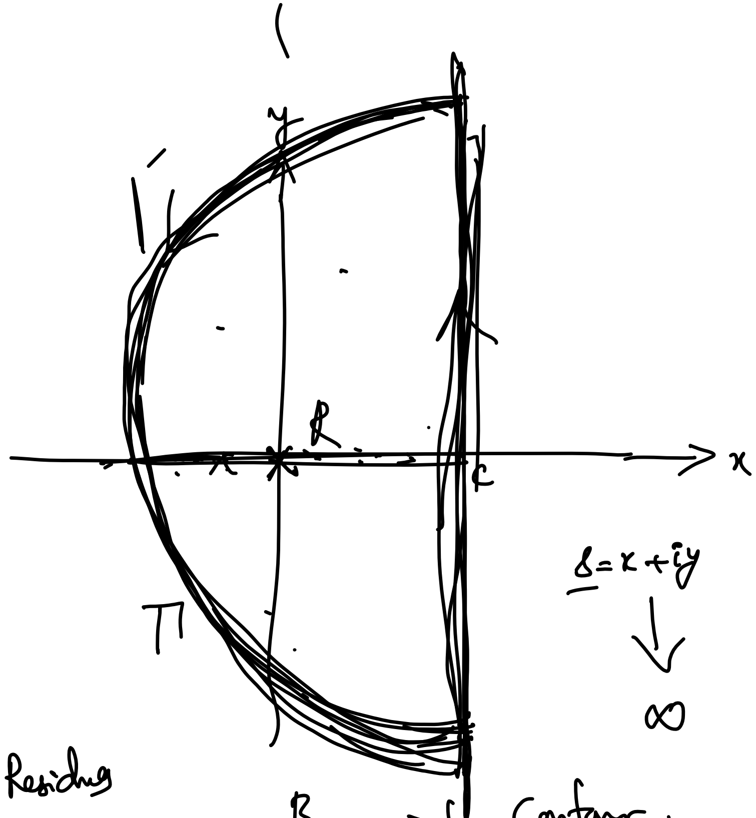
$c > a$ with 'a' being the exponential order of the function $f(t)$.

$$\mathcal{L}^{-1}\left(\bar{f}(s)\right)(t) = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(s)\, e^{st}\, ds.$$

$$= \sum_{k=0}^{\infty} \operatorname{Res}_{s=s_k}\left(\bar{f}(s)\, e^{st}\right)$$

$$\int_{\gamma} f(z)\, dz = 2\pi i \sum \text{Residu}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(s)\, e^{st}\, ds = \sum \text{Residue}$$



$s = x + iy$

$\downarrow$

∞

Bromwich Contour.

$$\boxed{\operatorname{Re}(s) < c}$$

Complex function theory:

$f : \mathbb{C} \longrightarrow \mathbb{C}$ ✓

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ ✓

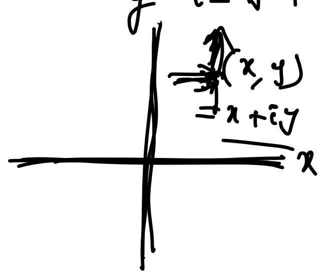
Analytic function: Let D be a domain
open connected set.

$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$, $z \in \mathbb{C}$ $\quad x+iy$

$f(z)$ is analytic at z_0

$f((x,y)) = \big(u(x,y),\ v(x,y)\big)$

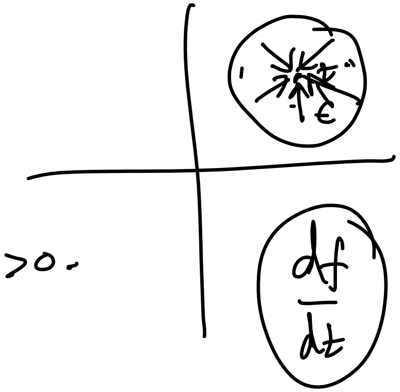
$i = \sqrt{-1}$



if $f(z)$ is differentiable

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

for all $z \in N_\epsilon(z_0)$, $\epsilon > 0$.



$$\frac{df}{d\vec{x}} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}, \quad \vec{x} = (x,y)$$

$f(z) = e^z$, z^2, z^n

Cauchy-Riemann equations:

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \iff f(z) = u + iv \text{ is analytic at } z.$$

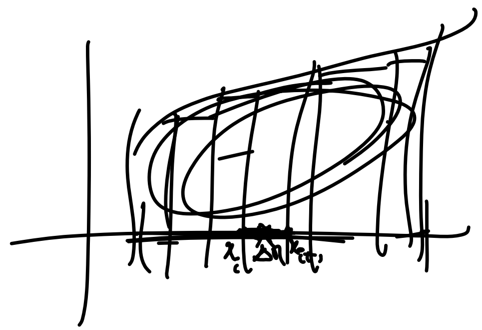
u_y, u_x, v_x, v_y all cts at (x,y)

Taylor Series

$f(z)$ is analytic at $z = z_0 \iff f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad z \in N_\epsilon(z_0), \quad \epsilon > 0$

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$; is analytic $z=0$

$$\int_a^b f(x)\,dx := \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$



Improper integral : $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx := \lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b f(x)\,dx$

$f : \mathbb{R} \longrightarrow \mathbb{C}$, $f(x) = u(x) + i\,v(x)$.

$$\int_a^b f(x)\,dx := \lim_{\Delta x_i\to 0} \sum_{i=1}^{n} \left(u(x_i^*) + i\,v(x_i^*)\right) \Delta x_i .$$

$$= \lim_{\Delta x_i\to 0} \sum_{i=1}^{n} u(x_i^*)\,\Delta x_i + i \lim_{\Delta x_i\to 0} \sum_{i=1}^{n} v(x_i^*)\,\Delta x_i$$

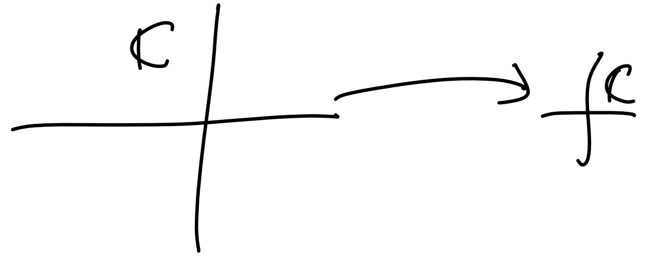
$$= \int_a^b u(x)\,dx + i\int_a^b v(x)\,dx$$

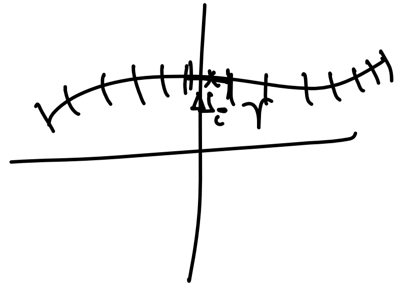
$$f : \mathbb{C} \longrightarrow \mathbb{C}$$

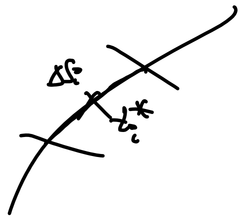
$$f(z) = \underline{u(x,y)} + i\, \underline{v(x,y)}. \qquad z = x + iy.$$

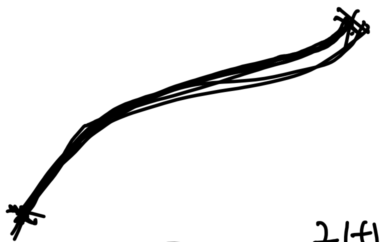
$$\underline{\int_\gamma f(z)\, dz} := \lim_{\Delta s_i \to 0} \sum_{i=1}^{n} f(z_i^*)\, \Delta s_i$$

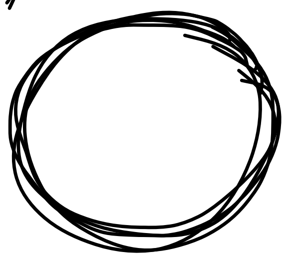
$$\underline{:= \int_a^b f\big(x(t) + i\, y(t)\big)\, z'(t) \cdot dt}$$











$$\underline{\gamma : [a, b] \longrightarrow \mathbb{C}}$$

$$z(t) = \underline{\gamma}(t) = x(t) + i\, y(t), \qquad t \in [a, b]$$

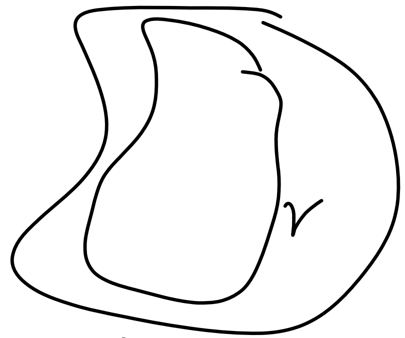
$$\frac{dz}{dt} = \frac{dx}{dt} + i \frac{dy}{dt}$$

$$x^2 + y^2 = 1$$

$$\underline{x = \cos t, \quad y = \sin t}$$

$$\underline{x^2 + y^2 = 1} \,/ \qquad \gamma(t) = \cos t + i \sin t, \quad t \in [0, 2\pi)$$

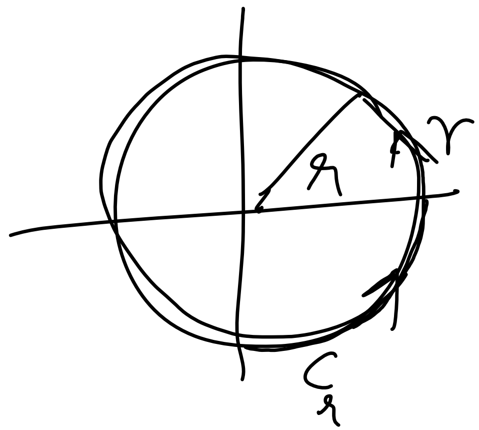
<u>Cauchy Theorem:</u> If $f(z)$ is analytic in a domain D. γ be a closed curve in D. Then



$$\oint_{\gamma} f(z)\,dz = 0, \quad \forall\ \gamma \subset D.$$

$z = re^{i\theta}$, $x = r\cos\theta$, $y = r\sin\theta$

γ is a closed circle of radius 'r'.



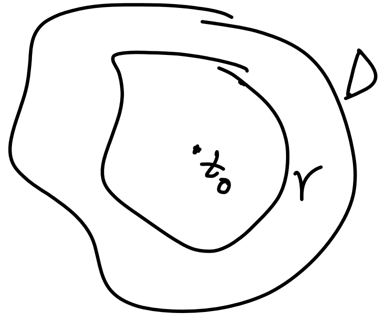
$$\oint_{C_r} \frac{dz}{z} = \int_0^{2\pi} \frac{\cancel{r}\, i\, \cancel{e^{it}}\, dt}{\cancel{r e^{it}}} = 2\pi i.$$

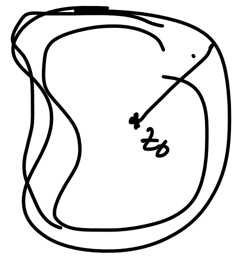
$z(t) = r\cos t + i\, r\sin t = re^{it}$.

$dz = z'(t)\,dt = r i e^{it}\,dt$

thm: If $f(z)$ is analytic in D then

$$\oint_\gamma \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0) \quad \text{(Cauchy integral formula).}$$



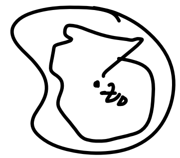


Cauchy residue theorem:

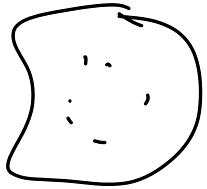
If $f(z)$ is analytic in D except at $z_0, z_1, z_2, \ldots, z_n$ in D and

then $$\oint_\gamma f(z)\,dz = 2\pi i \sum_{i=0}^{n} \operatorname*{Res}_{z=z_i} f(z)$$ $z_0, z_1, \ldots, z_n$ are inside γ.

$$\frac{f(z)}{z-z_0} = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-1}.$$



$$f(z) = \cdots + \frac{\cdot}{(z-z_0)^n} + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{z-z_0} + \left(\frac{c_0}{1!} + \frac{c_1(z-z_0)}{2!} + \cdots\right)$$



What type of integrals one can evaluate

1. $$\int_0^{2\pi} R(\sin\theta, \cos\theta)\, d\theta.$$

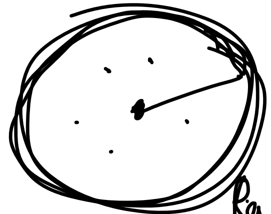
$$= \oint_{|z|=1} \frac{\tilde{R}(z, \bar{z})}{iz}\, dz$$

$$= 2\pi i \cdot \operatorname{Res}_{z=0} (\text{integrand})$$

$$z = e^{i\theta}$$

$$\frac{z+\bar{z}}{2} = \cos\theta, \quad \frac{z-\bar{z}}{2i} = \sin\theta$$

$$dz = i e^{i\theta}\, d\theta = iz\, d\theta$$



$$\operatorname{Res}_{z=0} \left(e^{1/z}\right) = 1$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} \cdot \frac{1}{2!} + \cdots$$

$$f(z) = \frac{C_{-m}}{(z-z_0)^m} + \cdots + \frac{C_{-1}}{z-z_0} + C_0 + C_1(z-z_0) + \cdots$$

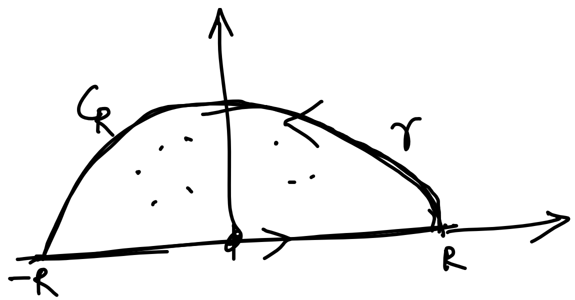
$$\forall\, z \in N_\epsilon(z_0)$$

$$C_{-1} = \operatorname{Res}_{z=z_0} f(z)$$

$$f(z)(z-z_0)^m = C_{-m} + \cdots + C_{-1}(z-z_0)^{m-1} + C_0(z-z_0)^m + \cdots$$

$$\frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[f(z)(z-z_0)^m\right] = C_{-1}$$

2. $$\int_{-\infty}^{\infty} f(x)\,dx := \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx .$$



$$\oint_{\gamma} f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname*{Res}_{z=z_j} f(z) \quad , \quad z_j \in \text{inside } \gamma .$$

If $\lim_{z\to\infty} z^2 f(z) = \text{Const.}$,

$$\text{L.H.S} = \int_{-R}^{R} f(x)\,dx + \cancel{\int_{C_R} f(z)\,dz} = 2\pi i \sum \text{Res}$$

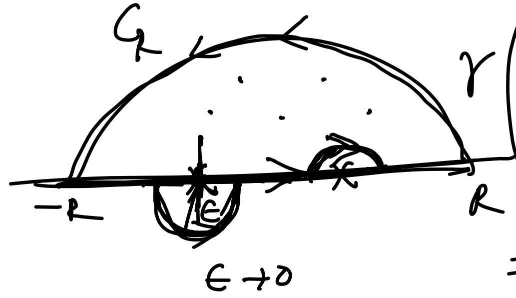
$\downarrow$ $\int_{-\infty}^{\infty} f(x)\,dx$

$\downarrow$ 0 as $R\to\infty$

$$f(z) = \frac{P(z)}{Q(z)}$$

3. $I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{iAx}\, dx$ ✓

$\deg(Q(x)) \geq 1 + \deg(P(x))$. ✓



$$\int_{-1}^{1} \frac{1}{x}\, dx$$

$$= \lim_{\epsilon \to 0} \int_{-1}^{-\epsilon} \frac{1}{x}\, dx + \int_{\epsilon}^{1} \frac{1}{x}\, dx$$

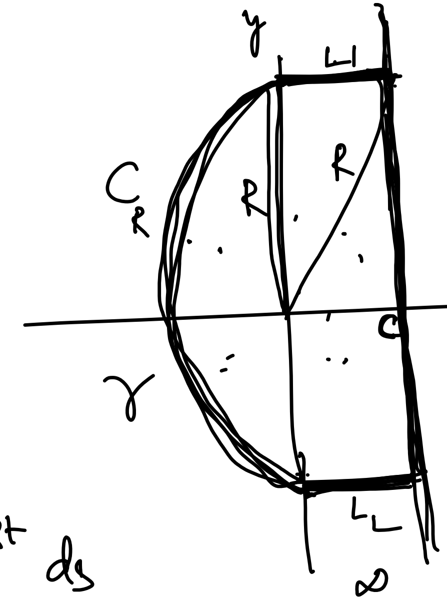
$$= \lim_{\epsilon \to 0} \log(-\epsilon) - \log(\epsilon)$$

$$\oint_{\gamma} \frac{P(z)}{Q(z)} e^{iAz}\, dz = \int_{-R}^{R} \frac{P(x)}{Q(x)} e^{iAx}\, dx + \int_{C_R} \frac{P(z)}{Q(z)} e^{iAz} = 2\pi i \sum_{j=1}^{n} \operatorname*{Res}_{z=z_j} \left(\frac{P(z)}{Q(z)} e^{iAz} \right)$$

$\int_{-R}^{R} \frac{P(x)}{Q(x)} e^{iAx}\, dx \to I$

$\int_{C_R} \frac{P(z)}{Q(z)} e^{iAz} \to 0$ as $R \to \infty$.

✓ $$f(t) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \bar{f}(s)\, e^{st}\, ds, \quad \text{Re}(s) > C > a$$



As $R\to\infty$, $L\to 0$

Re(s) < C

$s = x+iy$

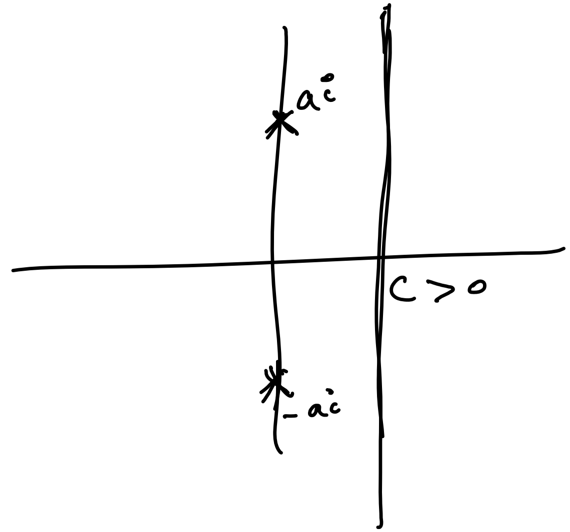
Consider

$$\lim_{R\to\infty} \oint_{\gamma} \bar{f}(s)\, e^{st}\, ds = \int_{C-i\infty}^{C+i\infty} + \cancel{\int_{L_1\cup L_2}} + \cancel{\int_{C_R} \bar{f}(s)\, e^{st}\, ds}$$

$$\bar{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt, \quad \text{Re}(s) > c$$

$$f(t) = \sum_{j=1}^{n} \text{Res}_{s=s_j}\left(\bar{f}(s)\, e^{st}\right)$$ ✓

Example: 1. $\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right)(t) = \cos at.$



$$\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right)(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{s}{s^2+a^2}\, e^{st}\, ds$$

$$= \operatorname*{Res}_{s=ai}\left(\frac{s}{s^2+a^2}\, e^{st}\right) + \operatorname*{Res}_{s=-ai}\left(\frac{s}{s^2+a^2}\, e^{st}\right).$$

$$= \lim_{s\to ai} \frac{s}{(s^2+a^2)}\, e^{st}\, \cancel{(s-ai)} + \lim_{s\to -ai} \frac{s}{(s^2+a^2)}\, e^{st}\, \cancel{(s+ai)}$$

$$= \lim_{s\to ai} \frac{s}{(s+ai)}\, e^{st} + \lim_{s\to -ai} \frac{s}{(s-ai)}\, e^{st}$$

$$= \frac{\cancel{ai}}{2\cancel{ai}} e^{ait} + \frac{\cancel{ai}}{2\cancel{ai}} e^{-ait}$$

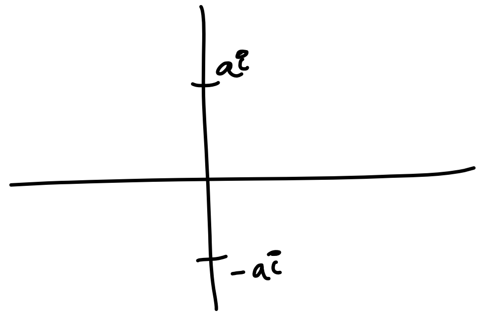
$$= \frac{1}{\cancel{2}} \left(\cancel{2} \cos at \right) = \cos at$$

2. $$\mathcal{L}^{-1} \left(\frac{s}{(s^2+a^2)^2} \right) = \frac{1}{a} \mathcal{L}^{-1} \left(\frac{a}{(s^2+a^2)} \cdot \frac{s}{(s^2+a^2)} \right) = \frac{1}{a} \mathcal{L}^{-1} \left(\mathcal{L}(\sin at) \cdot \mathcal{L}(\cos at) \right)$$

$$= \frac{1}{a} \cdot \int_0^t \sin a\tau \cos a(t-\tau) \, d\tau$$

$$= \frac{t}{2a} \sin at$$

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{s}{(s^2+a^2)^2} e^{st}\, ds$$



$$= \operatorname*{Res}_{s=ai} \frac{s e^{st}}{(s^2+a^2)^2} + \operatorname*{Res}_{s=-ai} \frac{s e^{st}}{(s^2+a^2)^2}.$$

$$= \lim_{s\to ai} \frac{d}{ds}\left(\cancel{(s-ai)^2}\cdot \frac{s e^{st}}{\cancel{(s-ai)^2}(s+ai)^2}\right) + \lim_{s\to -ai}\frac{d}{ds}\left(\cancel{(s+ai)^2}\,\frac{s e^{st}}{(s-ai)^2\cancel{(s+ai)^2}}\right)$$

$$= \frac{t\, e^{iat}}{4ia} - \frac{t\, e^{-iat}}{4ia} = \frac{t}{2a}\sin at$$

initial value theorem:

If $\bar{f}(s) = \mathcal{L}(f(t))(s)$, then $\lim\limits_{s \to \infty} \bar{f}(s) = 0$.

If $f'(t)$ exists, then $\lim\limits_{s \to \infty} [s\, \bar{f}(s)] = \lim\limits_{t \to 0} f(t) = f(0)$.

Proof:

$$\lim_{s \to \infty} \bar{f}(s) = \lim_{s \to \infty} \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty \underline{\lim_{s \to \infty} e^{-st}} \cdot f(t)\, dt$$

$$= \underline{0}$$

$$\lim_{s \to \infty} \mathcal{L}(f'(t)) = \lim_{s \to \infty} \int_0^\infty f'(t)\, e^{-st} dt = \int_0^\infty \lim_{s \to \infty} e^{-st}\, f'(t)\, dt = 0.$$

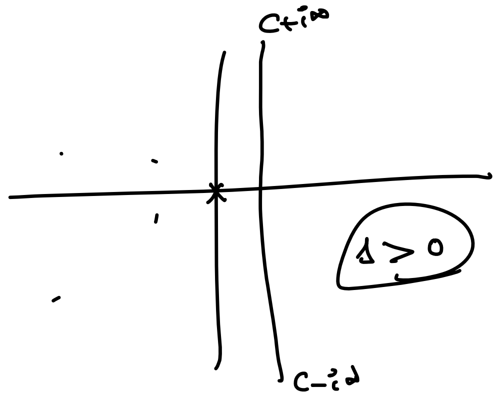
$$\lim_{s\to\infty} \left[s\bar{f}(s) - f(0) \right] = 0 .$$

$$\Rightarrow \lim_{s\to\infty} s\bar{f}(s) = f(0) = \lim_{t\to 0} f(t) \quad ✓$$

<u>Final value theorem</u>:

If $\bar{f}(s) = \dfrac{\bar{p}(s)}{\bar{q}(s)}$, where $\bar{p}(s)$ and $\bar{q}(s)$ are polynomials in s with $\deg(\bar{p}(s)) < \deg(\bar{q}(s))$.

Also, all roots of $\bar{q}(s) = 0$ have $-$ve real part, except possibly one root at $s = 0$,



$$\lim_{s\to 0} \bar{f}(s) = \int_0^\infty f(t)\, dt .$$

Also, if <u>$f'(t)$ exists</u>, $\lim_{s\to 0} \left(s\bar{f}(s) \right) = \lim_{t\to\infty} f(t) = f(\infty)$.

$\bar{f}(s)$, $s > 0$

Proof:

$$\lim_{s\to 0} \bar{f}(s) = \lim_{s\to 0} \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty \left[\lim_{s\to 0} e^{-st}\right] f(t)\, dt$$

$$= \int_0^\infty f(t)\, dt.$$

$$\lim_{s\to 0} \mathcal{L}(f'(t))(s) = \int_0^\infty f'(t)\, dt = f(\infty) - f(0).$$

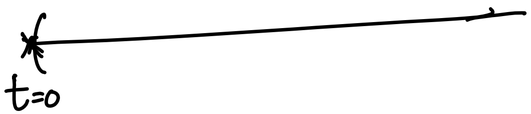
$$\lim_{s\to 0} \left[s\bar{f}(s) - \cancel{f(0)}\right] = f(\infty) - \cancel{f(0)} \Rightarrow \lim_{s\to 0} s\bar{f}(s) = f(\infty) = \lim_{t\to\infty} f(t)$$

Applications of Laplace transform:

Solutions of ordinary differential equations.

example: first order linear ODE.

$$\frac{dx}{dt} + p x = f(t), \quad t > 0.$$



$$x(0) = a.$$

where a, p are constants.

Soln: Apply Laplace transform to the equation, we get

$$s\,\bar{x}(s) - x(0) + p\,\bar{x}(s) = \bar{f}(s).$$

$$\Rightarrow \quad \bar{x}(s) = \frac{\bar{f}(s) + a}{s+p} \,..$$

Taking inverse transform, we get the solution

$$x(t) = a\,\mathcal{L}^{-1}\left(\frac{1}{s+p}\right) + \mathcal{L}^{-1}\left(\bar{f}(s)\cdot\frac{1}{s+p}\right)$$

$$= a\,e^{-pt} + \int_0^t f(\tau)\,e^{-p(t-\tau)}\,d\tau$$

$$\boxed{x(t) = a\,e^{-pt} + e^{-pt}\int_0^t f(\tau)\,e^{p\tau}\,d\tau.}$$ ✓

Second order linear ODE.

$$\frac{d^2x}{dt^2} + 2p\frac{dx}{dt} + qx = f(t), \quad t > 0$$

$x(0) = \underline{a}$, $\quad \frac{dx(0)}{dt} = x'(0) = \underline{b}$, where p, q, a, b are constants.

Soln: Apply L.T to the equation, we get

$$\left(s^2\bar{x}(s) - s\,\underline{x(0)} - x'(0)\right) + 2p\left(s\,\bar{x}(s) - \underline{x(0)}\right) + q\,\bar{x}(s) = \bar{f}(s).$$

$$\left(s^2 + 2ps + q\right)\bar{x}(s) = \bar{f}(s) + a(s+2p) + b$$

$$\Rightarrow \bar{x}(s) = \frac{\bar{f}(s) + a(s+2p) + b}{s^2 + 2ps + q}.$$

Inversion gives $\quad x(t) = \mathcal{L}^{-1}\left(\bar{f}(s)\cdot\frac{1}{s^2+2ps+q}\right) + a\,\mathcal{L}^{-1}\left(\frac{s+2p+b/a}{s^2+2ps+q}\right)$

$$= \mathcal{L}^{-1}\left(\frac{a(s+p)+(b+pa)+\bar{f}(s)}{(s+p)^2+n^2}\right) \qquad \underline{q-p^2=n^2} \left.\begin{matrix}>0\\<0\\=0\end{matrix}\right\}$$

$$x(t) = a\,\mathcal{L}^{-1}\left(\frac{s+p}{(s+p)^2+n^2}\right) + \frac{(b+pa)}{n}\,\mathcal{L}^{-1}\left(\frac{n}{(s+p)^2+n^2}\right) + \frac{1}{n}\mathcal{L}^{-1}\left(\bar{f}(s)\cdot\frac{n}{(s+p)^2+n^2}\right)$$

If $n^2 = q-p^2 > 0$,

$$x(t) = a\,e^{-pt}\cos nt + \frac{b+pa}{n}\left(e^{-pt}\sin nt\right) + \frac{e^{-pt}}{n}\int_0^t f(\tau)\,e^{p\tau}\sin n(t-\tau)\,d\tau$$

If $n^2=0$, $\quad x(t) = a\,e^{-pt} + (b+pa)\,t\,e^{-pt} + e^{-pt}\int_0^t f(\tau)\cdot(t-\tau)\,e^{p\tau}\,d\tau$. ✓

If $\underline{n^2 = q - p^2 < 0}$, $x(t) = a\, e^{-pt} \cosh nt + \frac{(b+pa)}{n} \cdot \left(e^{-pt} \sinh nt \right) + \frac{e^{-pt}}{n} \int_0^t f(\tau) \sinh n(t-\tau)\, e^{p\tau}\, d\tau .$

Higher order linear ODE:

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_n x = \phi(t), \quad t > 0$$

Initial values
$$\begin{cases} x(0) = x_0 \\ x'(0) = x_1 \\ \vdots \qquad \vdots \\ x^{(n-1)}(0) = x_{n-1} \end{cases}$$

Soln: L.T gives

$$\left(s^n\bar{x}(s) - s^{n-1}x_0 - s^{n-2}x_1 - \cdots - s\,x_{n-2} - x_{n-1}\right) + a_1\left(s^{n-1}\bar{x}(s) - s^{n-2}x_0 - \cdots - x_{n-2}\right)$$

$$+ a_2\left(s^{n-2}\bar{x}(s) - s^{n-3}x_0 - \cdots - x_{n-3}\right) + \cdots + a_{n-1}\left(s\bar{x}(s) - x_0\right) + a_n\bar{x}(s) = \bar{\phi}(s).$$

$$\bar{x}(s)\underbrace{\left(s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + s\,a_{n-1} + a_n\right)}_{\bar{q}_n(s)} = \bar{\phi}(s) + \underbrace{x_0\left(s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-1}\right) + \cdots + (s+a_1)x_{n-2} + x_{n-1}}_{\bar{P}_{n-1}(s)}$$

$$\bar{x}(s) = \frac{\bar{\phi}(s) + \bar{P}_{n-1}(s)}{\bar{q}_n(s)}$$

I.L.T: gives

$$\Rightarrow \quad x(t) = L^{-1}\left(\bar{\phi}(s)\cdot\frac{1}{\bar{q}_n(s)}\right) + L^{-1}\left(\frac{\bar{P}_{n-1}(s)}{\bar{q}_n(s)}\right); \quad t > 0.$$

eg: Solve $\frac{d^3x}{dt^3} + \frac{dx^2}{dt^2} - 6\frac{dx}{dt} = 0; \quad t > 0$

I.C's $\begin{cases} x(0) = 1 \\ x'(0) = 0 \\ x''(0) = 5 \end{cases}$.

Sol: Applying Laplace transform gives,

$$\bar{x}(s) = \frac{s^2+s-1}{(s^2+s-6)\,s}$$

Inversion gives, $x(t) = \mathcal{L}^{-1}\left(\frac{s^2+s-1}{s(s^2+s-6)}\right)$; $t > 0$.

$$= \mathcal{L}^{-1}\left(\frac{1}{6} \cdot \frac{1}{s} + \frac{1}{3} \frac{1}{s+3} + \frac{1}{2} \frac{1}{s-2} \right).$$

$$\boxed{x(t) = \frac{1}{6} + \frac{1}{3} e^{-3t} + \frac{1}{2} e^{2t}.}$$

Linear System of ODE's :

* Solve

$$\frac{dx_1}{dt} = a_{11} x_1 + a_{12} x_2 + b_1(t), \quad t > 0$$

$$\frac{dx_2}{dt} = a_{21} x_1 + a_{22} x_2 + b_2(t)$$

I. values: $x_1(0) = x_{10}$, $x_2(0) = x_{20}$

$$\Updownarrow$$

$$\frac{d}{dt}(X(t)) = A_{2\times 2}\, X(t) + B(t), \quad \text{where}$$

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}.$$

<u>I. value</u>: $\underline{X(0) = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}}$

<u>soln</u>: Application of Laplace transform to the equations gives

$$s\,\bar{x}_1(s) - x_{10} = a_{11}\,\bar{x}_1(s) + a_{12}\,\bar{x}_2(s) + \bar{b}_1(s)$$

$$s\,\bar{x}_2(s) - x_{20} = a_{21}\,\bar{x}_1(s) + a_{22}\,\bar{x}_2(s) + \bar{b}_2(s)$$

$$\Rightarrow \quad \begin{aligned} \bar{x}_1(s)\,(s - a_{11}) - a_{12}\,\bar{x}_2(s) &= x_{10} + \bar{b}_1(s) \\ -\bar{x}_1(s)\,a_{21} + (s - a_{22})\,\bar{x}_2(s) &= x_{20} + \bar{b}_2(s). \end{aligned}$$

$$\begin{pmatrix} s-a_{11} & -a_{12} \\ -a_{21} & s-a_{22} \end{pmatrix} \begin{pmatrix} \bar{x}_1(s) \\ \bar{x}_2(s) \end{pmatrix} = \begin{pmatrix} x_{10}+\bar{b}_1(s) \\ x_{20}+\bar{b}_2(s) \end{pmatrix}$$

$$\bar{x}_1(s) = \frac{\begin{vmatrix} x_{10}+\bar{b}_1(s) & -a_{12} \\ x_{20}+\bar{b}_2(s) & s-a_{22} \end{vmatrix}}{\begin{vmatrix} s-a_{11} & -a_{12} \\ -a_{21} & s-a_{22} \end{vmatrix}}, \quad \bar{x}_2(s) = \frac{\begin{vmatrix} s-a_{11} & x_{10}+\bar{b}_1(s) \\ -a_{21} & x_{20}+\bar{b}_2(s) \end{vmatrix}}{\begin{vmatrix} s-a_{11} & -a_{12} \\ -a_{21} & s-a_{22} \end{vmatrix}} \checkmark$$

Inversion gives $\underline{x_1(t) \text{ and } x_2(t)}$.

<u>example</u>: Solution of $\dfrac{dx(t)}{dt} = AX$, $X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

where $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$

equs are
$$\frac{dx_1}{dt} = x_2 \qquad x_1(0) = 0$$
$$\frac{dx_2}{dt} = -2x_1 + 3x_2 \qquad x_2(0) = 1$$

L.T gives,
$$s\,\bar{x}_1(s) = \bar{x}_2(s)$$
$$s\,\bar{x}_2(s) - 1 = -2\,\bar{x}_1(s) + 3\,\bar{x}_2(s)$$
$$\Leftrightarrow \begin{pmatrix} s & -1 \\ 2 & s-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\checkmark\ \bar{x}_1(s) = \frac{1}{s^2 - 3s + 2} = \frac{1}{s-2} - \frac{1}{s-1}$$

$$\checkmark\ \bar{x}_2(s) = \frac{s}{s^2 - 3s + 2} = \frac{2}{s-2} - \frac{1}{s-1}$$

I.L.T gives

$$\boxed{\begin{aligned} x_1(t) &= e^{2t} - e^{t} \\ x_2(t) &= 2e^{2t} - e^{t} \end{aligned}}$$

second order system of ODE's

Example:

System:

$$\frac{d^2x_1}{dt^2} - 3x_1 - 4x_2 = 0$$

$$\frac{d^2x_2}{dt^2} + x_1 + x_2 = 0 \quad , \quad t > 0$$

I.C's:

$$\boxed{\begin{aligned} x_1(0) &= 0 \qquad & \frac{dx_1(0)}{dt} &= 2 \\ x_2(0) &= 0 \qquad & \frac{dx_2(0)}{dt} &= 0. \end{aligned}}$$

$$X' = \underbrace{\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}}_{4\times 4}^{A} \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ \frac{dx_1}{dt} = y_1 \\ \frac{dx_2}{dt} = y_2 \end{bmatrix}}^{X}$$

Soln: Apply Laplace transform to the equations, we get

$$s^2 \bar{x}_1(s) - 2 - 3\bar{x}_1(s) - 4\bar{x}_2(s) = 0$$

$$s^2 \bar{x}_2(s) + \bar{x}_1(s) + \bar{x}_2(s) = 0.$$

$$\Rightarrow \quad \bar{x}_1(s)(s^2-3) - 4\bar{x}_2(s) = 2$$

$$\bar{x}_1(s) + (1+s^2)\bar{x}_2(s) = 0$$

$$\bar{x}_1(s) = \frac{2(1+s^2)}{(s^2-1)^2} = \frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}$$

$$\bar{x}_2(s) = \frac{-2}{(s^2-1)^2} = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{(s-1)^2} - \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{(s+1)^2}$$

Inverse L.T gives

$$x_1(t) = te^{t} + te^{-t} = t\left(e^{t} + e^{-t}\right) = 2t\cosh t$$ ✓

$$x_2(t) = \frac{1}{2}e^{t} - \frac{1}{2}te^{t} - \frac{1}{2}e^{-t} - \frac{1}{2}te^{-t}$$

$$= \frac{1}{2}\left(e^{t} - e^{-t}\right) - \frac{t}{2}\left(e^{t} + e^{-t}\right)$$

$$x_2(t) = \sinh t - t\cosh t$$ ✓

* Solve the Bessel equation

$$t\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + a^2 t\, x(t) = 0, \quad t > 0$$

$$x(0) = 1, \quad x'(0) = C$$

Soln: L.T gives

$$\mathcal{L}\left(t\frac{d^2x}{dt^2}\right) + \mathcal{L}\left(\frac{dx}{dt}\right) + a^2\mathcal{L}(t\,x(t)) = 0.$$

$$-\frac{d}{ds}\left(\mathcal{L}\left(\frac{d^2x}{dt^2}\right)\right) + \left(s\,\bar{x}(s) - 1\right) + a^2\left(-\frac{d}{ds}\left(\bar{x}(s)\right)\right) = 0$$

$$\Rightarrow -\frac{d}{ds}\left(s^2\bar{x}(s) - s - \underline{c}\right) + s\,\bar{x}(s) - 1 - a^2\frac{d\bar{x}(s)}{ds} = 0$$

$$\Rightarrow \quad 2s\,\bar{x}(s) + s^2\frac{d\bar{x}(s)}{ds} \cancel{-1} - s\,\bar{x}(s) \cancel{+1} + a^2\frac{d\bar{x}(s)}{ds} = 0$$

$$\Rightarrow \quad (s^2 + a^2)\frac{d\bar{x}(s)}{ds} + s\,\bar{x}(s) = 0.$$

$$\Rightarrow \int \frac{d\bar{x}(s)}{\bar{x}(s)} = \int -\frac{s}{(s^2+a^2)}\,ds + \log A \quad , \text{ where } \log A \text{ is integration Constant.}$$

$$\Rightarrow \log\left(\bar{x}(s)\right) = -\log\sqrt{(s^2+a^2)} + \log A$$

$$\Rightarrow \bar{x}(s) = \frac{A}{\sqrt{s^2+a^2}} \quad , \quad A \text{ is an arbitrary constant.}$$

$$\boxed{x(t) = A\,\mathcal{L}^{-1}\left(\frac{1}{\sqrt{s^2+a^2}}\right)(t)}$$

$$\underline{1} = x(0) = \lim_{t\to 0} x(t) = \lim_{s\to\infty} s\,\bar{x}(s). \quad \text{By initial value Theorem.}$$

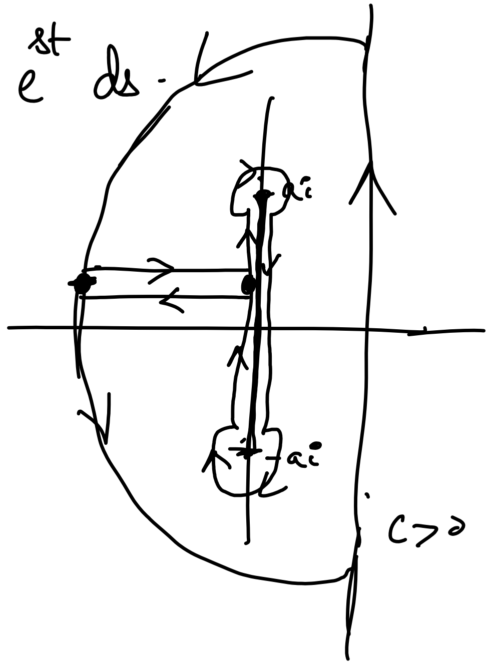
$$= \lim_{s\to\infty} \frac{s.A}{\sqrt{s^2+a^2}} = A.$$

$\Rightarrow \quad \underline{A = 1}$

$$\Rightarrow \quad x(t) = \mathcal{L}^{-1}\left(\frac{1}{\sqrt{s^2+a^2}}\right)(t).$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{\sqrt{s^2+a^2}}\, e^{st}\, ds.$$

$$= \underline{J_0(at)}$$



<u>Zero</u>

Zeroth order Bessel function of first kind:

$$J_0(at) := 1 - \frac{a^2t^2}{2^2} + \frac{a^4t^4}{2^2\cdot 4^2} - \frac{a^6t^6}{2^2\cdot 4^2\cdot 6^2} + \cdots$$

$$\mathcal{L}\left(J_0(at)\right) := \int_0^\infty e^{-st}\left(1 - \frac{a^2t^2}{2^2} + \frac{a^4t^4}{2^2\cdot 4^2} - \cdots\right) dt$$

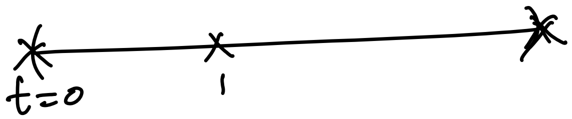
$$= \frac{1}{s} - \frac{a^2}{2^2}\cdot\frac{2!}{s^3} + \frac{a^4}{2^2\cdot 4^2}\cdot\frac{4!}{s^5} - \cdots$$

$$= \underline{\frac{1}{s}\left[1 - \frac{a^2}{2^2}\cdot\frac{2!}{s^2} + \frac{a^4}{2^2\cdot 4^2}\cdot\frac{4!}{s^4} - \cdots\right]}$$

$$= \frac{1}{s}\left(1 + \frac{a^2}{s^2}\right)^{-1/2}$$

$$= \frac{1}{\sqrt{s^2+a^2}}$$

* Solve $\frac{d^2x}{dt^2} + t\frac{dx}{dt} - 2x = 2, \quad t > 0.$



$x(t), \quad t > 0$

$x(0) = 0, \quad \frac{dx(0)}{dt} = 0.$

Soln: Apply L.T to the equation, we get

$$s^2\bar{x}(s) - \frac{d}{ds}\left(s\,\bar{x}(s)\right) - 2\bar{x}(s) = \frac{2}{s}$$

$$s^2\bar{x}(s) - \bar{x}(s) - s\frac{d\bar{x}(s)}{ds} - 2\bar{x}(s) = \frac{2}{s}$$

$$\Rightarrow \quad (s^2 - 3)\,\bar{x}(s) - s\frac{d\bar{x}(s)}{ds} = \frac{2}{s}$$

$$\Rightarrow \quad \frac{d\bar{x}}{ds} - \left(s - \frac{3}{s}\right)\bar{x}(s) = -\frac{2}{s^2}$$

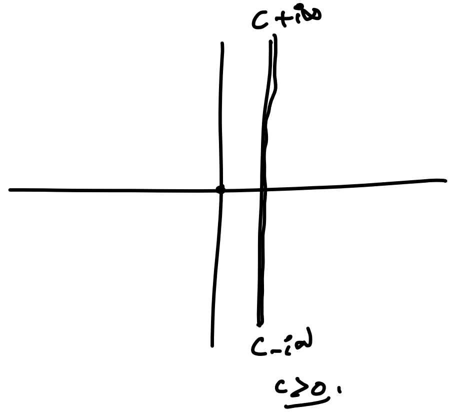
$$I.F = e^{-\int (s-\frac{3}{s})\,ds} = e^{-\frac{s^2}{2}+3\ln s} = s^3 e^{-\frac{s^2}{2}}.$$

$$\frac{d}{ds}\left(s^3 e^{-\frac{s^2}{2}}\,\bar{x}(s)\right) = -\frac{2}{s^2}\cdot s^3 e^{-s^2/2} = -2s\,e^{-s^2/2}.$$

$$\Rightarrow \quad \bar{x}(s)\cdot s^3 e^{-s^2/2} = -\int 2s\,e^{-s^2/2}\,ds + C$$

$$= 2e^{-s^2/2} + C$$

$$\Rightarrow \quad \bar{x}(s) = \frac{2}{s^3} + \frac{C}{s^3}e^{s^2/2}, \quad C \text{ is arbitrary constant.}$$



$$x(t) = t^2 + c\,\mathcal{L}^{-1}\left(\frac{e^{s^2/2}}{s^3}\right)(t).$$

$$0 = x(0) = \lim_{t\to 0} x(t) = \lim_{s\to\infty} s\,\bar{x}(s) = \lim_{s\to\infty}\left(\frac{2}{s^2} + \frac{c}{s^2}\, e^{s^2/2}\right)$$

$$0 = 0 + c\cdot \lim_{s\to\infty} \frac{e^{s^2/2}}{s^2}$$

$\Rightarrow$ $C = 0$

$$\Rightarrow \boxed{x(t) = t^2}$$

$$c_1\, x\Big|_{t=\pi} + c_2 \frac{dx}{dt}\Big|_{t=\pi} = 0$$

* Solve $\dfrac{d^2x(t)}{dt^2} + x(t) = t, \quad t > 0$

B.C's: $\dfrac{dx(0)}{dt} = 1, \quad x(\pi) = 0.$

Soln: L.T gives, $s^2\bar{x}(s) - s\,x(0) - 1 + \bar{x}(s) = \dfrac{1}{s^2}.$

Let $x(0) = A$.

$$(1+s^2)\,\bar{x}(s) = \frac{1}{s^2} + 1 + sA = \frac{1+s^2+s^3A}{s^2}.$$

$$\Rightarrow \quad \bar{x}(s) = \frac{1+s^2+s^3A}{s^2(1+s^2)} = \frac{1}{s^2} + \frac{sA}{1+s^2}.$$

$$L^{-1}\left(\frac{s}{s^2+1}\right) = \underline{\cos t}.$$

I.L.T gives

$$\Rightarrow \quad x(t) = t + A\cdot\cos t$$

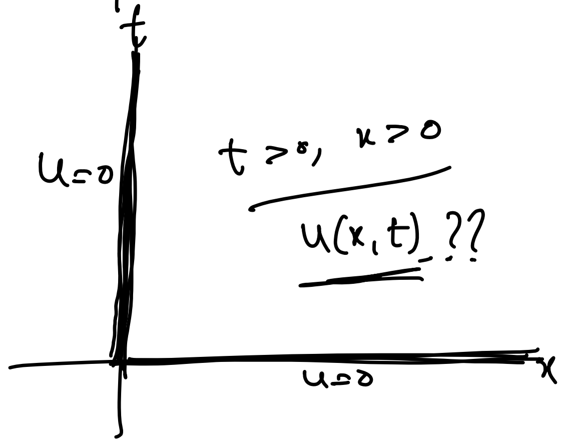
$$0 = x(\pi) = \pi - A \quad \Rightarrow \quad A = \pi. \checkmark$$

$$\Rightarrow \quad \boxed{x(t) = t + \pi\cos t} \checkmark$$

Partial differential equations: $F\left(u(x,t),\ u_x,\ u_t\right)=0$ more than one independent variable, one dependent variable.

(initial boundary value problem for first order PDE)

Example: Solve $u_t + x\,u_x = x, \quad x>0,\ t>0.$



I.C: $u(x,0)=0, \quad x>0$ ✓

B.C: $u(0,t)=0, \quad t>0$

Soln: Apply L.T to the equation w.r.to the variable 't', we get

$$s\,\bar{u}(x,s) - \cancel{u(x,0)} + x\cdot\frac{\partial}{\partial x}\left(\bar{u}(x,s)\right) = x\cdot\frac{1}{s}.$$

$$\Rightarrow \quad \frac{\partial \bar{u}}{\partial x} + \frac{s}{x}\,\bar{u} = \frac{1}{s}, \quad x>0$$

$$I.F = e^{\int \frac{s}{x}\,dx} = e^{s \ln x} = e^{\ln x^s} = x^s .$$

$$\frac{d}{dx}\left(x^s \, \bar{u}(x,s) \right) = \frac{x^s}{s} .$$

$$\Rightarrow x^s \, \bar{u}(x,s) = \int \frac{x^s}{s}\,dx + C(s) ,$$ C(s) is integration function . .

$$\Rightarrow \bar{u}(x,s) = \frac{x^{-s}}{s} \cdot \frac{x^{s+1}}{s+1} + C(s)\, x^{-s} .$$

$$x^s \, \bar{u}(x,s) = \frac{x^{s+1}}{s \cdot s+1} + \cancel{C(s)}$$

Since $\underline{u(0,t) = 0}$, $t>0$ $\Rightarrow$ $\bar{u}(0,s) = 0$ ✓

$$0 = \bar{u}(0,s) = C(s)$$

$$\bar{u}(x,s) = \frac{x}{s\cdot(s+1)} = \frac{x}{s} - \frac{x}{s+1}$$

Inversion gives, $\boxed{u(x,t) = x - xe^{-t} = x(1-e^{-t}), \; x>0,\ t>0}$ ✓

<u>Example:</u> Solve $x\, u_t + u_x = x, \quad x>0, \quad t>0$

<u>I.C:</u> $u(x,0) = 0, \quad x>0$ ✓

<u>B.C:</u> $u(0,t) = 0, \quad t>0.$

<u>Soln:</u> L.T w.r.to 't', makes the equation into

$$x\left(s\,\bar{u}(x,s)\right) + \frac{\partial \bar{u}(x,s)}{\partial x} = \frac{x}{s}.$$

$$\Rightarrow \quad \frac{\partial \bar{u}}{\partial x} + x s\,\bar{u} = \frac{x}{s}., \quad x > 0$$

$$I.F = e^{\int x s\, dx} = \underline{e^{s\frac{x^2}{2}}}$$

$$\frac{\partial}{\partial x}\left(\bar{u}(x,s)\cdot e^{s\frac{x^2}{2}}\right) = e^{s\frac{x^2}{2}}\cdot\frac{x}{s}.$$

$$\Rightarrow \quad \bar{u}(x,s)\, e^{s\frac{x^2}{2}} = x\int \frac{e^{s\frac{x^2}{2}}}{s}\,dx + C(s),$$ $C(s)$ is arbitrary function.

$$= \frac{e^{\frac{sx^2}{2}}}{s^2} + C(s)$$

$$\Rightarrow \quad \bar{u}(x,s) = \frac{1}{s^2} + C(s)\cdot e^{-s\frac{x^2}{2}}. \checkmark$$

B.C: $u(0,t)=0 \Rightarrow \underline{\bar{u}(0,s)=0}$ ✓

$$0 = \frac{1}{s^2} + C(s) \Rightarrow C(s) = -\frac{1}{s^2}.$$

$$\Rightarrow \quad \bar{u}(x,s) = \frac{1}{s^2}\left(1 - e^{-s\frac{x^2}{2}}\right).$$

$$\Rightarrow \quad u(x,t) = t - \mathcal{L}^{-1}\left(\frac{e^{-s\frac{x^2}{2}}}{s^2}\right)(t).$$

If $\mathcal{L}(f(t)) = \bar{f}(s)$, then $\mathcal{L}\left(f(t-a)\,H(t-a)\right) = e^{-as}\,\mathcal{L}(f(t))$

$$\left(\left(t-\frac{x^2}{2}\right) H\left(t-\frac{x^2}{2}\right)\right) = \mathcal{L}^{-1}\left(e^{-\frac{x^2}{2}s}\,\frac{1}{s^2}\right) \qquad f(t) = t$$

$$\Rightarrow \quad u(x,t) = t - \left[\left(t-\frac{x^2}{2}\right) H\left(t-\frac{x^2}{2}\right)\right], \qquad t>0,\ x>0.$$

$$\boxed{u(x,t) = \begin{cases} t, & 2t < x^2 \\ \dfrac{x^2}{2} & 2t \geq x^2 \end{cases}}, \quad t>0,\ x>0.$$

2^{nd} order linear PDE :

general eqn : $A\, u_{xx} + B\, u_{xy} + C\, u_{yy} + D u_x + E u_y + F u = G.$ ✓

$$B^2 - 4AC > 0 \longrightarrow \text{Hyperbolic} \longrightarrow u_{\xi\xi} = u_{\eta\eta} + \cancel{F(u_\xi, u_\eta, u)} = 0 \ ✓ \quad \text{Wave equation}$$

$(x, y) \xrightarrow{\text{Transformation}} (\xi, \eta)$

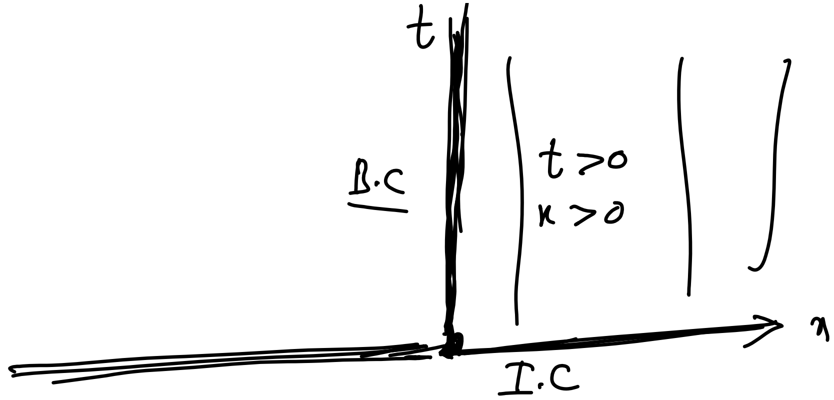
$$= 0 \longrightarrow \text{parabolic} \longrightarrow u_{\xi\xi} - u_\eta = 0 \ ✓ \quad \text{Heat equation}$$

$$< 0 \longrightarrow \text{elliptic} \longrightarrow u_{\alpha\alpha} + u_{\beta\beta} = 0, \quad \text{Laplace equation.}$$

$$u_{xy} + F(u_x, u_y, u) = 0 \quad \text{Hyperbolic}$$

$u(\xi, 0)$ I.C

B.C



Hyperbolic equations:

* Solve $u_{xt} = -w \sin wt, \quad t > 0; \; x \in \mathbb{R}$.

I.C: $u(x,0) = x$ ✓

B.C: $u(0,t) = 0$.

Soln: Application of Laplace transform to the equation w.r.to 't' variable gives

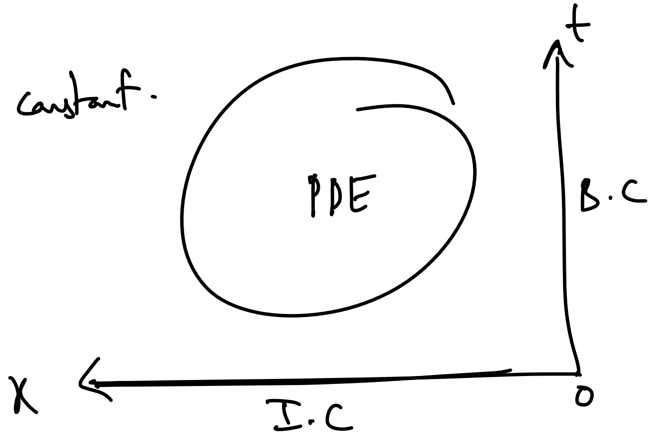
$$\frac{\partial}{\partial x}\left(s\,\bar{u}(x,s) - x\right) = -w \frac{w}{s^2+w^2}, \qquad x > 0$$

$$\cancel{s}\,\frac{\partial \bar{u}(x,s)}{\partial x} = 1 - \frac{w^2}{s^2+w^2} = \frac{s^{\cancel{2}}}{s^2+w^2}$$

$$\Rightarrow \quad \frac{\partial \bar{u}}{\partial x} = \frac{s}{s^2+w^2}.$$

Since $\quad u(0,t)=0, \quad \bar{u}(0,s)=0.$

$$\bar{u}(x,s) = \frac{s}{s^2+w^2}\, x + c; \quad c \text{ is constant.}$$



$$0 = \bar{u}(0,s) = c$$

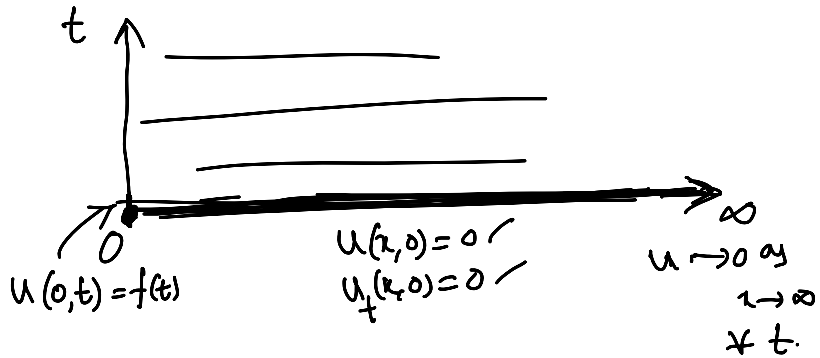
$$\Rightarrow \quad \bar{u}(x,s) = \frac{s x}{s^2+w^2}.$$

Inverse transform gives the solution

$$\boxed{u(x,t) = x \cdot \cos wt, \quad x>0,\ t>0}$$

Ily, we get $u(x,t) = x \cos wt \quad x<0,\ t>0$

$$\Rightarrow \quad u(x,t) = x\cos\omega t, \quad \forall\, x\in\mathbb{R},\ t>0.$$



* Transverse vibrations of a Semi-infinite string:

$$u_{tt} = c^2 u_{xx}, \quad x>0,\ t>0,\ c\neq 0$$

I.C: $u(x,0) = 0$

$\frac{\partial u}{\partial t}(x,0) = 0$

B.C: $u(0,t) = f(t)$

$u(x,t) \to 0$ as $x\to\infty$.

Soln: L. transform w.r.to 't' gives

$$s^2 \bar{u}(x,s) - s\cancel{u(x,0)} - \cancel{\frac{\partial u(x,0)}{\partial t}} = c^2 \frac{\partial^2 \bar{u}(x,s)}{\partial x^2}.$$

$$\Rightarrow \left.\begin{aligned} &\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{s^2}{c^2}\bar{u} = 0, \; x>0 \\ &\bar{u}(0,s) = \bar{f}(s). \checkmark \\ &\bar{u}(x,s) \to 0 \text{ as } x \to \infty. \end{aligned}\right\}$$

$y'' - a^2 y = 0$

$y(x) = c_1 e^{ax} + c_2 e^{-ax}.$ ✓

$$\bar{u}(x,s) = \cancel{c_1 e^{\frac{s}{c}x}} + c_2 e^{-\frac{s}{c}x}. \checkmark$$

Since $\bar{u} \to 0$ as $x \to \infty$, we have $c_1 = 0$ ✓.

$$\bar{f}(s) = \bar{u}(0,s) = c_2$$

$$\Rightarrow \quad \bar{u}(x,s) = \bar{f}(s)\, e^{-\frac{sx}{c}}, \quad x>0.$$

$$f(t-a)\,H(t-a) = \mathcal{L}^{-1}\left(e^{-as}\,\mathcal{L}(f(t))\right) \checkmark$$

Inversion gives the solution

$$u(x,t) = \mathcal{L}^{-1}\left(e^{-\frac{x}{c}s}\,\mathcal{L}(f(t))\right) = f\left(t-\frac{x}{c}\right)\cdot H\left(t-\frac{x}{c}\right).$$

$$\Rightarrow \quad u(x,t) = \begin{cases} 0, & t<\frac{x}{c} \\ f\left(t-\frac{x}{c}\right) & t \geq \frac{x}{c} \text{ or } x < tc \end{cases}$$

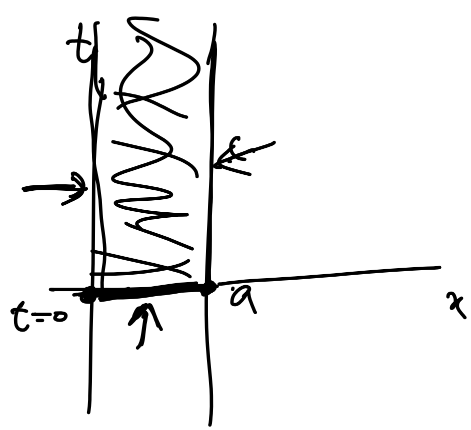
Inhomogeneous wave equation.

* Solve $\quad u_{tt} - c^2 u_{xx} = k \sin\left(\frac{\pi x}{a}\right), \quad 0<x<a, \quad t \geq 0.$

c - speed, k, a are constants.

$\underline{I.C:}$ $\quad u(x,0) = 0$ ✓

$\quad\quad\quad \dfrac{\partial u}{\partial t}(x,0) = 0$ ✓



$\underline{B.C's:}$ $\quad u(0,t) = 0$ ✓

$\quad\quad\quad u(a,t) = 0$ ✓

$\underline{\text{Soln}}$: L.T to the 't' variable gives

$$s^2\,\bar{u}(x,s) - \cancel{s\,u(x,0)} - \cancel{\frac{\partial u}{\partial t}(x,0)} - c^2\,\frac{\partial^2 \bar{u}(x,s)}{\partial x^2} = \underline{k \sin\left(\frac{\pi x}{a}\right)}\,\frac{1}{s}.$$

$$\Rightarrow \begin{cases} \dfrac{\partial^2 \bar{u}}{\partial x^2} - \dfrac{s^2}{c^2}\,\bar{u} = -\dfrac{k}{sc^2}\sin\left(\dfrac{\pi x}{a}\right), & 0 < x < a \\[2ex] \bar{u}(0,s) = 0, \quad \bar{u}(a,s) = 0 & \end{cases}$$

$$\bar{u}(x,s) = \underline{C_1}\, e^{\frac{s}{c}x} + \underline{C_2}\, e^{-\frac{s}{c}x} + \frac{k}{s}\cdot\frac{\sin\frac{\pi x}{a}}{\left(\frac{\pi^2 c^2}{a^2} + s^2\right)}, \quad x>0 \qquad m^2 - \frac{s^2}{c^2} = 0, \quad m = \pm\frac{s}{c}$$

$$\left(D^2 - \frac{s^2}{c^2}\right) u = \underline{\sin ax}$$

$$u_p = \frac{\sin ax}{(-a)^2 - \frac{s^2}{c^2}}$$

$$0 = \bar{u}(0,s) = C_1 + C_2 \ ✓$$

$$0 = \bar{u}(a,s) = C_1\, e^{\frac{s}{c}a} + C_2\, e^{-\frac{s}{c}a} + \cancel{\frac{k}{s}\,\frac{\sin\pi}{s^2+c^2a^2}} \ ✓$$

$$C_1\left(e^{\frac{s}{c}a} - e^{-\frac{s}{c}a}\right) = 0 \Rightarrow \underline{C_1 = 0} \ ✓$$

$$\Rightarrow \quad C_1 = C_2 = 0$$

$$\bar{u}(x,s) = \frac{k}{s}\,\frac{\sin\frac{\pi x}{a}}{s^2 + \left(\frac{\pi c}{a}\right)^2}; \quad x>0$$

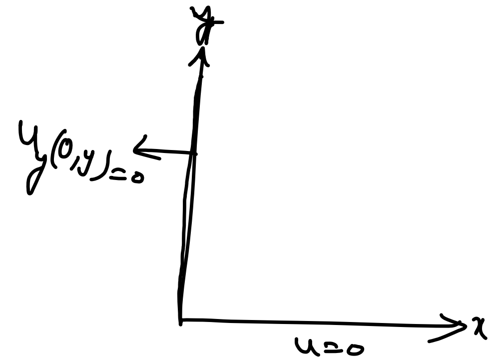
Inversion gives $\quad u(x,t) = k \sin\frac{\pi x}{a}\, L^{-1}\left(\frac{1}{s\left(s^2 + \left(\frac{\pi c}{a}\right)^2\right)}\right)$

$$= \frac{a^2}{\pi^2 c^2}\, k \sin\frac{\pi x}{a} \cdot L^{-1}\left(\frac{1}{s} - \frac{s}{s^2 + \left(\frac{\pi c}{a}\right)^2} \right)$$

$$\boxed{u(x,t) = \frac{k a^2}{\pi^2 c^2} \sin\frac{\pi x}{a} \cdot \left(1 - \cos\frac{\pi c}{a} t\right), \quad x > 0,\ t > 0}$$

* solve $\dfrac{\partial^2 u}{\partial x\, \partial y} = e^{-y} \cos x, \quad x > 0,\ y > 0.$

$u(x,0) = 0,\ x > 0; \quad \dfrac{\partial u(0,y)}{\partial y} = 0, \quad y > 0.$



Soln: Apply L.T to the equation w.r.to 'y', to get

$$\frac{\partial}{\partial x}\left(s\,\bar{u}(x,s) - \cancel{u(x,0)}\right) = \cos x \cdot \frac{1}{s+1}$$

$$\frac{\partial \bar{u}(x,s)}{\partial x} = \cos x \cdot \frac{1}{s(s+1)} \Rightarrow \frac{\partial u(x,y)}{\partial x} = L^{-1}\left(\cos x\left[\frac{1}{s} - \frac{1}{s+1}\right]\right)$$

$$\boxed{\frac{\partial u}{\partial x} = \cos x \cdot \left(1 - e^{-y}\right)}$$

This is a PDE of order 1.

Again, apply L.T w.r.t 'x', to see that

$$s\,\bar{u}(s,y) - u(0,y) = \frac{\cancel{s}}{s^2+1} \cdot \left(1 - e^{-y}\right).$$

$$\Rightarrow \bar{u}(s,y) = \frac{1}{s}\,u(0,y) + \left(1 - e^{-y}\right) \cdot \frac{1}{s(s^2+1)}$$

Inversion gives

$$\Rightarrow u(x,y) = u(0,y) + \left(1 - e^{-y}\right) L^{-1}\left(\frac{1}{s(s^2+1)}\right).$$

$$\Rightarrow \boxed{u(x,y) = \cancel{u(0,y)} + (1-e^{-y})(1-\cos x)}$$

$$\frac{1}{s} - \frac{s}{s^2+1}$$

$$0 = \left.\frac{\partial u(x,y)}{\partial y}\right|_{x=0} = \frac{\partial u(0,y)}{\partial y} \Rightarrow \frac{\partial u(0,y)}{\partial y} = 0$$

$$u(0,y) = f(x)$$

Soln:

Let $\frac{\partial u}{\partial y} = U$

$$\begin{cases} \frac{\partial U}{\partial x} = e^{-y}\cos x; & x>0, \quad y>0. \\ U(0,y) = 0 \end{cases}$$

Apply L.T to the equation w.r.to x, we get

$$\cancel{s}\, U(s,y) - \cancel{U(0,y)} = e^{-y} \cdot \frac{\cancel{s}}{s^2+1}$$

$$\Rightarrow \quad U(s,y) = \frac{e^{-y}}{s^2+1}.$$

Inversion gives $\quad U(x,y) = e^{-y} \cdot \sin x.$

$$\begin{cases} \dfrac{\partial u}{\partial y} = e^{-y} \sin x . \\ u(x,0) = 0 \end{cases}$$

Again, apply L.T to the above equation, to get

$$s\, u(x,s) - \cancel{u(x,0)} = \sin x \cdot \frac{1}{s+1}$$

$$\Rightarrow \quad u(x,s) = \sin x \, \frac{1}{s(s+1)} = \sin x . \left(\frac{1}{s} - \frac{1}{s+1} \right)$$

Inversion gives $\boxed{u(x,y) = \sin x \left(1 - e^{-y}\right).}$ ✓.

* Solve $\dfrac{\partial^2 u}{\partial x \partial y} = \sin x \sin y$; $x > 0,\ y > 0$

B.Cs : $u(x,0) = 1 + \cos x$ ✓

$u_y(0,y) = -2 \sin y$ ✓

Soln: Let $V(x,y) = \dfrac{\partial u(x,y)}{\partial y}$.

$$\left.\begin{aligned} &\frac{\partial V}{\partial x} = \sin x \sin y \\ &\frac{\partial u(0,y)}{\partial y} = V(0,y) = -2 \sin y. \end{aligned}\right\}$$

L.T gives w.r.to 'x' $\Big\}$ $s\,\bar{U}(s,y) - U(0,y) = \sin y \cdot \dfrac{1}{s^2+1}$

$$\Rightarrow s\,\bar{U}(s,y) + \underline{2\sin y} = \sin y \cdot \frac{1}{s^2+1}.$$

$$\Rightarrow s\,\bar{U}(s,y) = \sin y\left(\frac{1}{s^2+1} - 2\right).$$

$$\Rightarrow \bar{U}(s,y) = \sin y\,\frac{1-2s^2-2}{(s^2+1)\,s}.$$

$$\bar{U}(s,y) = -\sin y \cdot \frac{2s^2+1}{s(s^2+1)} \;✓.$$

Inversion gives $U(x,y) = \dfrac{\partial u}{\partial y} = -\sin y\,\mathcal{L}^{-1}\left[\dfrac{2s^{\cancel{2}}}{\cancel{s}(s^2+1)} + \underline{\dfrac{1}{s(s^2+1)}}\right]$.

$$= -\sin y \, L^{-1}\left[\frac{2s}{s^2+1} + \frac{1}{s} - \frac{s}{s^2+1}\right]$$

$$= -\underline{\sin y \cdot 2 \cdot \cos x} - \sin y + \underline{\sin y \cos x}$$

$$\frac{\partial u}{\partial y} = -\sin y \cos x - \sin y \checkmark$$

Application of L.T w.r.to 'y' gives,

$$s\,\bar{u}(x,s) - \underline{u(x,0)} = -(1+\cos x)\frac{1}{s^2+1}$$

$$\Rightarrow s\,\bar{u}(x,s) = -(1+\cos x)\frac{1}{s^2+1} + (1+\cos x)$$

$$s\,\bar{u}(x,s) = \left(\frac{-1}{s^2+1} + 1\right)(1+\cos x).$$

$$\bar{u}(x,s) = \frac{s}{s^2+1}(1+\cos x).$$

Invert to get $\boxed{u(x,y) = (1+\cos x)\cos y.}$ $\underline{x>0,\ y>0}$

Heat equation :

* Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \quad 0<x<2, \quad t>0.$

I.C : $u(x,0) = 3\sin(2\pi x).$

B.C's: $u(0,t)=0, \ \forall\, t>0$

$u(2,t)=0, \ \forall\, t>0$

Soln: we apply Laplace transform to the equation w.r.to 't', to see that

$$s\,\bar{u}(x,s) - \underline{u(x,0)} = \frac{\partial^2 \bar{u}}{\partial x^2}, \quad 0 < x < 2.$$

$$\Rightarrow \quad \frac{\partial^2 \bar{u}}{\partial x^2} - s\,\bar{u}(x,s) = -3\sin(2\pi x), \quad 0 < x < 2.$$

$m^2 - s = 0$

$m = \pm\sqrt{s}.$

$\underline{(D^2 + a^2)\,y = \sin kx}$

$$y_p(x) = \frac{\sin kx}{-k^2 + a^2}$$

L.T to the B.C's gives, $\bar{u}(0,s) = 0$

$\bar{u}(2,s) = 0.$

$$\bar{u}(x,s) = c_1 e^{\sqrt{s}\,x} + c_2 e^{-\sqrt{s}\,x} + \frac{3\sin(2\pi x)}{4\pi^2 + s}$$

$$\left.\begin{aligned} 0 &= \bar{u}(0,s) = c_1 + c_2 \\ 0 &= \bar{u}(2,s) = c_1 \underline{e^{2\sqrt{s}}} + c_2 \underline{e^{-2\sqrt{s}}} \end{aligned}\right\} \quad \underline{c_1 = c_2 = 0},$$

$$\Rightarrow \quad \bar{u}(x,s) = \frac{3\sin(2\pi x)}{4\pi^2+s}$$

$$\begin{bmatrix} 1 & 1 \\ e^{2\sqrt{s}} & e^{-2\sqrt{s}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$c_1 = c_2 \neq 0$

$e^{-2\sqrt{s}} - e^{2\sqrt{s}} \neq 0$

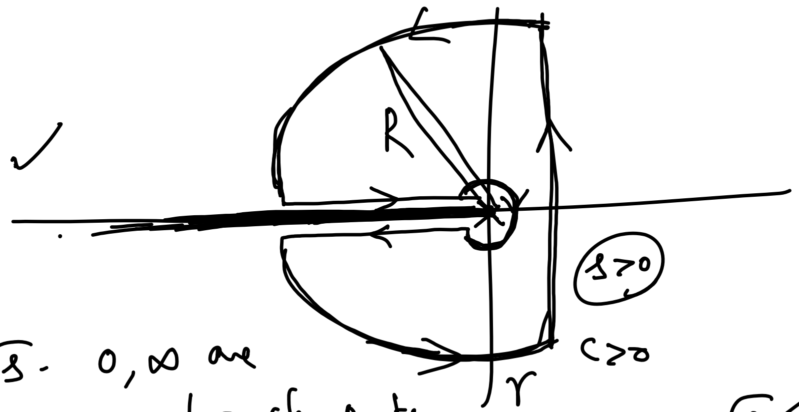
Inversion of L.T gives $\boxed{u(x,t) = 3\sin(2\pi x)\cdot e^{-4\pi^2 t}, \quad 0<x<2, \; t>0}$ ✓

$$\mathcal{L}^{-1}\left(\frac{1}{s+4\pi^2}\right) = e^{-4\pi^2 t}$$

Remark: If $c_1 = c_2 \neq 0$ for non-zero B.C's, we need

$$\mathcal{L}^{-1}\left(e^{-a\sqrt{s}}\right)(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-a\sqrt{s}}\, e^{st}\, ds \; ✓$$

$a>0$

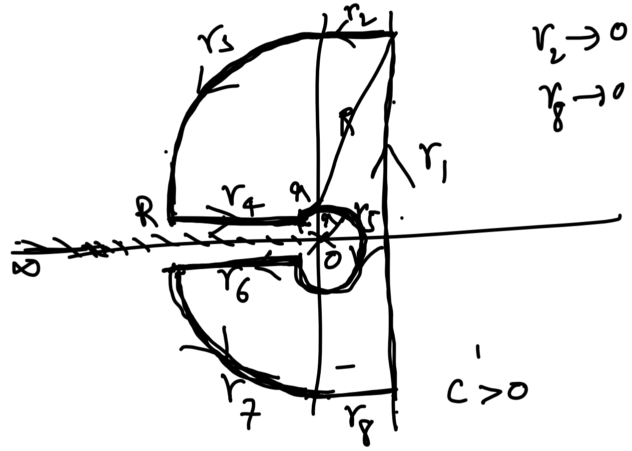


$\sqrt{s}$ - $0, \infty$ are branch points

$$= \frac{a}{t}\frac{1}{\sqrt{4\pi t}}\, e^{-a^2/4t}, \quad a>0.$$

Laplace inversion of $e^{-a\sqrt{s}}$, $a > 0$.

$$\mathcal{L}^{-1}\left(e^{-a\sqrt{s}}\right)(t) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-a\sqrt{s}}\, e^{st}\, ds, \quad c$$



$\gamma_2 \to 0$ as $R \to \infty$
$\gamma_8 \to 0$ as $R \to \infty$

$c > 0$

$$\gamma = \bigcup_{i=1}^{8} \gamma_i$$

Consider $$\int_{\gamma} e^{-a\sqrt{s}}\, e^{st}\, ds = 0.$$

$\because$ inside γ, $e^{-a\sqrt{s}}\, e^{st}$ is analytic

$$\Rightarrow \int_{\gamma_1} + \cancel{\int_{\gamma_3}} + \int_{\gamma_4} + \cancel{\int_{\gamma_5}} + \int_{\gamma_6} + \cancel{\int_{\gamma_7}} e^{-a\sqrt{s}}\, e^{st}\, ds = 0.$$

$$\Rightarrow \int_{\gamma_3+\gamma_7} e^{-a\sqrt{s}} e^{st}\, ds \longrightarrow 0 \text{ as } R\to\infty.$$

γ_5 : $s = re^{i\theta}$ $0<\theta<2\pi$

$$\int_{\gamma_5} e^{-a\sqrt{s}} e^{st}\, ds = \int_0^{2\pi} e^{-a\sqrt{r}\,e^{i\theta/2}} e^{re^{i\theta}t}\, ire^{i\theta}\, d\theta \longrightarrow 0 \text{ as } r\to 0$$

$$\Rightarrow \int_{\gamma_1} + \int_{\gamma_4} + \int_{\gamma_6} e^{-a\sqrt{s}} e^{st}\, ds = 0.$$

$\sqrt{s} = \sqrt{r}\, e^{i\theta/2}$, $-\pi<\theta<\pi$

$r>0$

γ_7 : $s = Re^{i\theta}$, $-\pi<\theta<-\pi/2$ $ds = Rie^{i\theta}d\theta$

$$\int_{\pi/2}^{\pi} \left| e^{-a\sqrt{R}\,e^{i\theta/2}} e^{Re^{i\theta}} Rie^{i\theta} \right| d\theta$$

$-\pi<\theta<-\pi/2$

$\cos\theta/2 > 0$ ✓

$\cos\theta < 0$, ✓

$$\left. \begin{array}{l} e^{-a\sqrt{R}\cos\theta/2} \\ e^{R\cos\theta\, t} \end{array} \right\} \to 0 \text{ as } R\to\infty$$

$$\int_{\gamma_4} e^{-a\sqrt{s}} e^{st}\, ds \longrightarrow \int_0^\infty e^{-xt} e^{-a\sqrt{x}\, i}\, dx, \quad \text{as } r \to 0,\ R \to \infty,$$

γ_4: $s = x e^{i\pi}$, $r < x < R$
$ds = -dx$.

$$\int_{\gamma_4} e^{-a\sqrt{s}} e^{st}\, ds = -\int_R^r e^{-a\sqrt{x}\, i} e^{-xt}\, dx$$

$$\Rightarrow \int_{\gamma_6} e^{-a\sqrt{s}} e^{st}\, ds \longrightarrow -\int_0^\infty e^{-xt} e^{a\sqrt{x}\, i}\, dx, \quad \text{as } r \to 0,\ R \to \infty.$$

γ_6: $s = x e^{-i\pi}$, $r < x < R$.
$ds = -dx$

$$\int_{\gamma_6} e^{-a\sqrt{s}} e^{st}\, ds = -\int_r^R e^{a\sqrt{x}\, i} e^{-xt}\, dx.$$

$$\int_{\gamma_1} e^{-a\sqrt{s}} e^{st}\, ds \longrightarrow \int_{c-i\infty}^{c+i\infty} e^{-a\sqrt{s}}, e^{st}\, ds \quad \text{as } r \to 0,\ R \to \infty$$

$$\mathcal{L}^{-1}\left(e^{-a\sqrt{s}}\right)(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-a\sqrt{s}}\, e^{st}\, ds = \frac{1}{2\pi i}\left[\int_0^\infty e^{-xt}\, e^{ia\sqrt{x}}\, dx - \int_0^\infty e^{-xt}\, e^{-ia\sqrt{x}}\, dx\right]$$

$$I = \mathcal{L}^{-1}\left(e^{-a\sqrt{s}}\right)(t) = \frac{1}{\pi}\int_0^\infty e^{-xt} \sin(a\sqrt{x})\, dx.$$

Let $x = u^2$ $\quad dx = 2u\, du$

$$I = \frac{2}{\pi}\int_0^\infty u e^{-u^2 t} \sin(au)\, du = \sqrt{\frac{2}{\pi}}\, \mathcal{F}_s\left(u e^{-u^2 t}\right)(a).$$

$$\mathcal{L}^{-1}\left(e^{-\xi\sqrt{s}}\right)(t) = \sqrt{\frac{2}{\pi}}\, \mathcal{F}_s\left(x e^{-x^2 t}\right)(\xi) = \frac{2}{\pi}\int_0^\infty x e^{-x^2 t} \sin \xi x\, dx$$

$$\mathcal{F}\left(x e^{-x^2 t}\right)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2 t} e^{-i\xi x}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\infty} x e^{-x^2 t} \underline{e^{-i\xi x}\, dx} - \int_0^{\infty} x e^{-x^2 t} \underline{e^{i\xi x}}\, \underline{dx} \right]$$

$$= \frac{-2i}{\sqrt{2\pi}} \int_0^{\infty} x e^{-x^2 t} \sin \xi x \, dx.$$

$$= -i \underline{\sqrt{\frac{2}{\pi}} \int_0^{\infty} x e^{-x^2 t} \sin \xi x \, dx} = -i\, \underline{\mathcal{F}_s\left(x e^{-x^2 t}\right)(\xi)}.$$

$$\Rightarrow \mathcal{L}^{-1}\left(e^{-\xi\sqrt{s}}\right)(t) = i\sqrt{\frac{2}{\pi}} \cdot \mathcal{F}\left(x e^{-x^2 t}\right)(\xi) = \underline{i\sqrt{\frac{2}{\pi}}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2 t} e^{-i\xi x}\, dx.$$

To evaluate

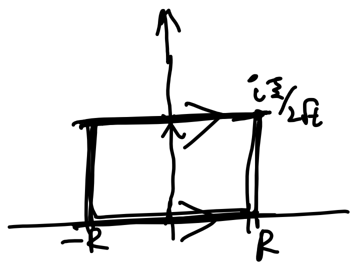
$$\mathcal{F}\left(x e^{-x^2 t}\right)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2 t} e^{-i\xi x}\, dx .$$

We first evaluate $\mathcal{F}\left(e^{-x^2 t}\right)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underline{e^{-x^2 t} e^{-i\xi x}}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x\sqrt{t} + \frac{i\xi}{2\sqrt{t}}\right)^2} \cdot e^{-\frac{\xi^2}{4t}}\, dx .$$

$$= e^{-\frac{\xi^2}{4t}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x\sqrt{t} + \frac{i\xi}{2\sqrt{t}}\right)^2} \underline{dx} .$$

$$= \frac{e^{-\frac{\xi^2}{4t}}}{\sqrt{t}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2}\, dx$$



$$x\sqrt{t} + \frac{i\xi}{2\sqrt{t}} = x_1 \checkmark$$

$$\underline{dx\,\sqrt{t} = dx_1} .$$

$$e^{-\left[\underline{(x\sqrt{t})^2 + 2x\sqrt{t}\,b + \underline{b^2}}\right]} e^{-\frac{\xi^2}{4t}}$$

$$2\cancel{x}\sqrt{t}\,b = i\xi\cancel{x}$$

$$\Rightarrow b = i\frac{\xi}{2\sqrt{t}} .$$

$$b^2 = -\frac{\xi^2}{4t}$$

$$= \frac{e^{-\frac{\xi^2}{4t}}}{\sqrt{t}} \cdot \frac{1}{\sqrt{2\pi}} \sqrt{\pi} = \frac{1}{\sqrt{2t}} e^{-\frac{\xi^2}{4t}} \checkmark$$

$$\Rightarrow \quad \mathcal{F}\left(e^{-x^2 t}\right)(\xi) = \frac{1}{\sqrt{2t}} \cdot e^{-\frac{\xi^2}{4t}}, \quad \xi \in (-\infty, \infty).$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2 t} e^{-i\xi x} \, dx = \frac{1}{\sqrt{2t}} e^{-\frac{\xi^2}{4t}}, \quad \xi \in \mathbb{R}.$$

$$-i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2 t} e^{-i\xi x} \, dx = \frac{1}{\sqrt{2t}} e^{-\frac{\xi^2}{4t}} \cdot \left(-\frac{\cancel{2}\xi}{\underset{2}{\cancel{4}}t}\right) = -\frac{1}{t\, 2\sqrt{2}\sqrt{t}} e^{-\frac{\xi^2}{4t}}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-x^2 t}\, e^{-i\xi x}\, dx = -i \frac{\xi}{t\, 2\, \sqrt{2t}}\, e^{-\frac{\xi^2}{4t}} \checkmark$$

$$\boxed{\mathcal{F}\left(x e^{-x^2 t}\right)(\xi) = -\frac{i\xi}{2}\sqrt{\frac{1}{2t}}\,\frac{1}{t}\, e^{-\frac{\xi^2}{4t}}.}\ , \quad t>0. \checkmark$$

$$\Rightarrow \mathcal{L}^{-1}\left(e^{-\xi\sqrt{s}}\right)(t) = -i\sqrt{\frac{2}{\pi}}\,\frac{i\xi}{2\, t}\sqrt{\frac{1}{2t}}\, e^{-\frac{\xi^2}{4t}}$$

$$= \frac{\xi}{2\sqrt{\pi}\sqrt{t}\sqrt{t^2}}\, e^{-\frac{\xi^2}{4t}}, \quad t>0.$$

$$\mathcal{L}^{-1}\left(e^{-\xi\sqrt{s}}\right)(t) = \frac{\xi}{\sqrt{4\pi t^3}}\, e^{-\frac{\xi^2}{4t}}, \quad t>0$$

$$\therefore \quad \boxed{\mathcal{L}^{-1}\left(e^{-a\sqrt{s}}\right)(t) = \frac{a}{\sqrt{4\pi t^3}}\, e^{-\frac{a^2}{4t}}, \quad a>0}$$

Laplace inversion of $\dfrac{e^{-a\sqrt{s}}}{s}$. ie, $\mathcal{L}^{-1}\left(\dfrac{e^{-a\sqrt{s}}}{s}\right)(t)$.

$$\frac{a}{2\sqrt{\tau}} = x$$

$$-\frac{a}{2}\cdot\frac{1}{2}\cdot \tau^{-3/2}\, d\tau = dx.$$

$$\mathcal{L}^{-1}\left(\frac{e^{-a\sqrt{s}}}{s}\right)(t) = \mathcal{L}^{-1}\left(\mathcal{L}(1)\cdot \mathcal{L}\left(\frac{a}{\sqrt{4\pi t^3}}\, e^{-a^2/4t}\right)\right)$$

$$= \int_0^t \frac{a}{\sqrt{4\pi \tau^3}}\, e^{-\frac{a^2}{4\tau}}\, d\tau.$$

$$= -\int_{\infty}^{a/2\sqrt{t}} \frac{\cancel{a}}{\sqrt{4\pi}\,\cancel{\tau^{3/2}}}\, e^{-x^2}\, \frac{4\,\cancel{\tau^{3/2}}\, dx}{\cancel{a}} = \frac{2}{\sqrt{\pi}}\int_{a/2\sqrt{t}}^{\infty} e^{-x^2}\, dx$$

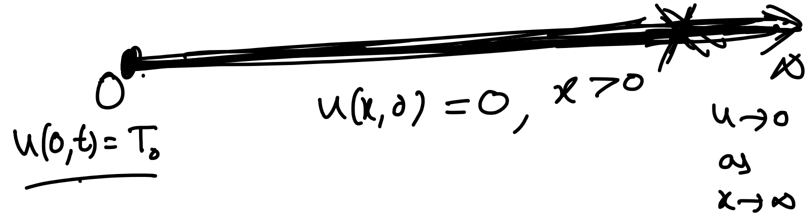
$$\boxed{\mathcal{L}^{-1}\left(\frac{e^{-a\sqrt{s}}}{s}\right)(t) = \frac{2}{\sqrt{\pi}}\int_{a/2\sqrt{t}}^{\infty} e^{-x^2}\,dx, \quad a>0}$$

Differentiate both sides w.r.t 'a', to get

$$+\mathcal{L}^{-1}\left(\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right)(t) = +\frac{\cancel{2}}{\sqrt{\pi}}\frac{1}{\cancel{2}\sqrt{t}}\cdot e^{-\frac{a^2}{4t}}$$

$$\boxed{\mathcal{L}^{-1}\left(\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right)(t) = \frac{1}{\sqrt{\pi t}}\, e^{-\frac{a^2}{4t}}, \quad a>0}$$

Heat conduction in a semi-infinite rod:



* Solve $u_t = k\, u_{xx}$, $x > 0$, $t > 0$

I.C: $u(x, 0) = 0$

B.C's: $u(0, t) = f(t)$ ✓

$u(x, t) \to 0$ as $x \to \infty$ ✓

Soln: Application of Laplace transform to the equation w.r.to 't' gives

$$s\,\bar{u}(x,s) - \cancel{u(x,0)} = k \frac{\partial^2 \bar{u}(x,s)}{\partial x^2}, \quad x > 0.$$

$$\Rightarrow \frac{\partial^2 \bar{u}}{\partial x^2} - \frac{s}{k}\bar{u} = 0, \quad x > 0. ✓$$

$$\bar{u}(0,s) = \bar{f}(s) \checkmark$$

$$\bar{u}(x,s) \to 0 \text{ as } x \to \infty.$$

$$m^2 - \frac{s}{k} = 0$$

$$m = \pm\sqrt{\frac{s}{k}}.$$

$$\bar{u}(x,s) = \cancel{c_1 e^{\sqrt{\frac{s}{k}}x}} + c_2 e^{-\sqrt{\frac{s}{k}}x}$$

Since $\bar{u}(x,s) \to 0$ as $x \to \infty$, $c_1 = 0$.

$$\bar{u}(x,s) = c_2 e^{-\sqrt{\frac{s}{k}}x}.$$

$$\bar{f}(s) = \bar{u}(0,s) = c_2$$

$$\Rightarrow \bar{u}(x,s) = \bar{f}(s).e^{-\sqrt{\frac{s}{k}}\cdot x}, \quad x > 0.$$

Inversion gives $\boxed{u(x,t) = \int_0^t f(t-\tau) \frac{\frac{x}{\sqrt{k}}}{\sqrt{4\pi\tau^3}} e^{-\frac{x^2}{4k\tau}} d\tau, \quad t > 0, x > 0}$

Remark: If $f(t) = T_0$

$$u(x,t) = \frac{x\, T_0}{\sqrt{k4\pi}} \int_0^t \frac{e^{-\frac{x^2}{4k\tau}}}{\tau\sqrt{\tau}}\, d\tau .$$

$$\frac{x}{2\sqrt{k\tau}} = x_1 \Rightarrow -\frac{1}{2}\frac{x}{2\sqrt{k}}\tau^{-3/2}\, d\tau = dx_1$$

$$\frac{d\tau}{\tau\sqrt{\tau}} = -\frac{4\sqrt{k}}{x}\, dx_1$$

$$= \frac{-\cancel{x}\; T_0}{\sqrt{\cancel{4k}\pi}} \int_\infty^{x/2\sqrt{kt}} \frac{2\cancel{4}\cancel{\sqrt{k}}}{\cancel{x}} \cdot e^{-x_1^2}\, dx_1$$

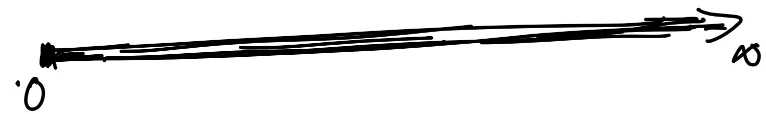
$$u(x,t) = \frac{2T_0}{\sqrt{\pi}} \int_{x/2\sqrt{kt}}^{\infty} e^{-\tau^2}\, d\tau$$

As $t \to \infty$, $u(x,t) = \frac{2T_0}{\sqrt{\pi}} \cdot \int_0^\infty e^{-\tau^2}\, d\tau = T_0$; $x \geq 0$.

* Solve $\quad u_t = k\, u_{xx}, \quad x>0, \quad t>0.$

<u>I.C:</u> $\quad u(x,0) = 0$

<u>B.C's:</u>
$$\begin{cases} \dfrac{\partial u(0,t)}{\partial x} = g(t), & t>0 \\[2ex] u(x,t) \to 0 \text{ as } x\to\infty. \end{cases}$$



$$\frac{\partial u(0,t)}{\partial x} = g(t).$$

<u>Solve:</u> L.T gives

$$\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{s}{k}\bar{u} = 0; \quad x>0$$

$$\frac{\partial \bar{u}(0,s)}{\partial x} = \bar{g}(s), \qquad \bar{u}(x,s) \to 0 \text{ as } x\to\infty.$$

$$\underline{\bar{u}(x,s) = C_2}\, e^{-\sqrt{\frac{s}{k}}\,x}, \qquad x > 0$$

$$\bar{g}(s) = \frac{\partial \bar{u}(0,s)}{\partial x} = -C_2 \cdot \sqrt{\frac{s}{k}} \Rightarrow C_2 = -\sqrt{\frac{k}{s}} \cdot \bar{g}(s).$$

$$\boxed{\mathcal{L}^{-1}\left(\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right)(t) = \frac{e^{-\frac{a^2}{4t}}}{\sqrt{\pi t}}}$$

$$\Rightarrow \quad \bar{u}(x,s) = -\sqrt{k} \cdot \underbrace{\bar{g}(s)}_{g(t)} \cdot \frac{e^{-\frac{x}{\sqrt{k}}\cdot\sqrt{s}}}{\sqrt{s}}, \qquad x > 0.$$

Invention gives

$$u(x,t) = -\sqrt{k} \int_0^t g(t-\tau) \frac{1}{\sqrt{\pi\tau}} e^{-\frac{x^2}{4k\tau}} \, d\tau.$$

$$\boxed{u(x,t) = -\sqrt{\frac{k}{\pi}} \int_0^t g(t-\tau) \frac{1}{\sqrt{\tau}} e^{-\frac{x^2}{4k\tau}} \, d\tau.}, \quad x \geq 0, \ t > 0.$$

Remark: If $g(t) = T_0$ = constant,

$$u(x,t) = -\sqrt{\frac{k}{\pi}}\, T_0 \cdot \int_0^t \frac{e^{-\frac{x^2}{4k\tau}}}{\sqrt{\tau}}\, d\tau \, ,$$

$$\frac{x}{2\sqrt{k\tau}} = x_1 \Rightarrow -\frac{1}{2}\frac{x}{2\sqrt{k}}\tau^{-3/2}\, d\tau = dx_1$$

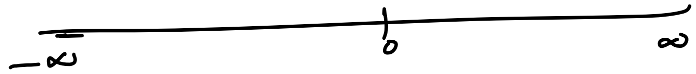
$$\frac{x^2}{4k\tau} = x_1^2$$

$$\tau = \frac{x^2}{x_1^2 4k}.$$

$$= +\sqrt{\frac{\cancel{k}}{\pi}}\, T_0 \cdot \int_\infty^{x/2\sqrt{kt}} e^{-x_1^2}\, \frac{\cancel{4}\, dx_1 \cancel{\sqrt{k}}}{\cancel{x}}\, \frac{x^{\cancel{2}}}{x_1^2 \cancel{4k}}.$$

$$\boxed{u(x,t) = -\sqrt{\frac{1}{\pi}}\, T_0 x \int_{x/2\sqrt{kt}}^{\infty} \tau^{-2} e^{-\tau^2}\, d\tau \, , \quad x > 0,\ t \geq 0}$$

* Solve $u_t = k\, u_{xx}$, $x \in \mathbb{R}$, $t > 0$



I.C: $u(x,0) = f(x)$ ✓

B.C: $u(x,t) \to 0$ as $|x| \to \infty$. ✓

Soln: L.T gives $s\,\bar{u}(x,s) - f(x) = k \dfrac{\partial^2 \bar{u}(x,s)}{\partial x^2}$

$$\Rightarrow \begin{cases} \dfrac{\partial^2 \bar{u}}{\partial x^2} - \dfrac{s}{k}\bar{u} = -\dfrac{f(x)}{k}, & x \in (-\infty, \infty). \\ \bar{u}(x,s) \to 0 & \text{as } x \to \infty \\ \bar{u}(x,s) \to 0 & \text{as } x \to -\infty. \end{cases}$$

$$\bar{u}(x,s) = \underbrace{c_1 e^{\sqrt{\frac{s}{k}}\,x}}_{y_1} + \underbrace{c_2 e^{-\sqrt{\frac{s}{k}}\,x}}_{y_2} + \int_0^x \frac{\left[-y_1(x)\,y_2(\xi) + y_2(x)\,y_1(\xi)\right]}{W(\xi)} \left(\frac{-f(\xi)}{k}\right) d\xi$$

$$W(y_1,y_2)(x) = \begin{vmatrix} e^{\sqrt{s/k}\,x} & e^{-\sqrt{s/k}\,x} \\ \sqrt{s/k}\, e^{\sqrt{s/k}\,x} & -\sqrt{\frac{s}{k}}\, e^{-\sqrt{s/k}\,x} \end{vmatrix}$$

$$= -\sqrt{\frac{s}{k}} - \sqrt{s/k} = -\frac{2}{\sqrt{k}}\sqrt{s}.$$

$$\bar{u}(x,s) = c_1 e^{\sqrt{s/k}\,x} + c_2 e^{-\sqrt{s/k}\,x} - \frac{\sqrt{k}}{2\sqrt{s}\,k} e^{\sqrt{s/k}\,x} \int_0^x e^{-\sqrt{s/k}\,\xi} f(\xi)\, d\xi$$

$$+ \frac{\sqrt{k}}{2\sqrt{s}\,k} e^{-\sqrt{s/k}\,x} \int_0^x e^{\sqrt{s/k}\,\xi} f(\xi)\, d\xi$$

$$\bar{u}(x,s) = \underline{c_1 e^{\sqrt{s/k}\,x}} + \underline{c_2 e^{-\sqrt{s/k}\,x}} - \frac{1}{2\sqrt{s}\sqrt{k}} \underline{e^{\sqrt{s/k}\,x}} \int_0^x f(\xi)\, e^{-\sqrt{s/k}\,\xi} d\xi + \frac{1}{2\sqrt{s}\sqrt{k}} \underline{e^{-\sqrt{s/k}\,x}} \int_0^x e^{\sqrt{s/k}\,\xi} f(\xi)\, d\xi.$$

$$\bar{u}(x,s) = \left(c_1 - \frac{1}{2\sqrt{s}\sqrt{k}} \int_0^x f(\xi)\, \underline{e^{-\sqrt{s/k}\,\xi}}\, d\xi\right) e^{\sqrt{s/k}\,x} + \left(c_2 + \frac{1}{2\sqrt{s}\sqrt{k}} \int_0^x \underline{f(\xi)\, e^{\sqrt{s/k}\,\xi}}\, d\xi\right) e^{-\sqrt{s/k}\,x}. \checkmark$$

Since $\bar{u}(x,s) \to 0$ as $x \to \infty$, $$C_1 = \frac{1}{2\sqrt{s}\sqrt{k}} \underline{\int_0^\infty f(\xi)\, e^{-\sqrt{s/k}\,\xi}\, d\xi} \quad \checkmark$$

Again, $\bar{u}(x,s) \to 0$ as $x \to -\infty$, $$C_2 = \frac{1}{2\sqrt{s}\sqrt{k}} \int_{-\infty}^0 f(\xi)\, e^{\sqrt{s/k}\,\xi}\, d\xi .$$

$$\Rightarrow \bar{u}(x,s) = \frac{1}{2\sqrt{s}\sqrt{k}} \int_x^\infty \underline{f(\xi)}\, \underline{e}^{-\sqrt{s/k}(\xi - x)}\, d\xi + \frac{1}{2\sqrt{s}\sqrt{k}} \int_{-\infty}^x f(\xi)\, \underline{e}^{-\sqrt{s/k}(x-\xi)}\, d\xi .$$

$\underline{(\xi - x) > 0}$

$\xi < x$

$x - \xi = -(\xi - x) > 0 .$

$$\bar{u}(x,s) = \frac{1}{2\sqrt{s}\sqrt{k}} \int_{-\infty}^\infty f(\xi)\, e^{-\sqrt{s/k}|\xi - x|}\, d\xi$$

Inversion of Laplace transform gives

$$u(x,t) = \frac{1}{2\sqrt{k}} \int_{-\infty}^{\infty} f(\xi)\, \mathcal{L}^{-1}\left(e^{-\frac{|\xi - x|}{\sqrt{k}}\sqrt{s}} \right)(t)\, d\xi$$

$$= \frac{1}{2\sqrt{k}} \int_{-\infty}^{\infty} f(\xi)\, \frac{e^{-\frac{(\xi-x)^2}{4kt}}}{\sqrt{\pi t}} .$$

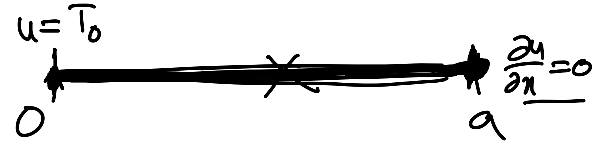
$$\boxed{u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} f(\xi)\, e^{-\frac{(x-\xi)^2}{4kt}}\, d\xi , \quad x \in \mathbb{R},\ t > 0}$$

* Solve $u_t = k\, u_{xx}, \quad 0 < x < a, \quad t > 0 .$

I.C: $u(x,0) = 0, \quad 0 < x < a$

B.C's: $u(0,t) = T_0, \quad t > 0.$

$\dfrac{\partial u(a,t)}{\partial x} = 0, \quad t > 0.$



Soln: L.T gives

$$\begin{cases} \dfrac{d^2 \bar{u}}{dx^2} - \dfrac{s}{k}\bar{u} = 0; \quad 0 < x < a \\[2ex] \bar{u}(0,s) = \dfrac{T_0}{s}. \\[2ex] \dfrac{\partial \bar{u}}{\partial x}(a,s) = 0. \end{cases}$$

$$\bar{u}(x,s) = C_1 e^{\sqrt{s/k}\,x} + C_2 e^{-\sqrt{s/k}\,x}.$$

$$\bar{u}(0,s) = \frac{T_0}{s} = c_1 + c_2$$

$$\frac{\partial \bar{u}(a,s)}{\partial x} = 0 = \sqrt{s/k}\left(c_1 e^{\sqrt{s/k}\,a} - c_2 e^{-\sqrt{s/k}\,a}\right)$$

$$c_1 e^{\sqrt{s/k}\,a} - \left(\frac{T_0}{s} - c_1\right) e^{-\sqrt{s/k}\,a} = 0.$$

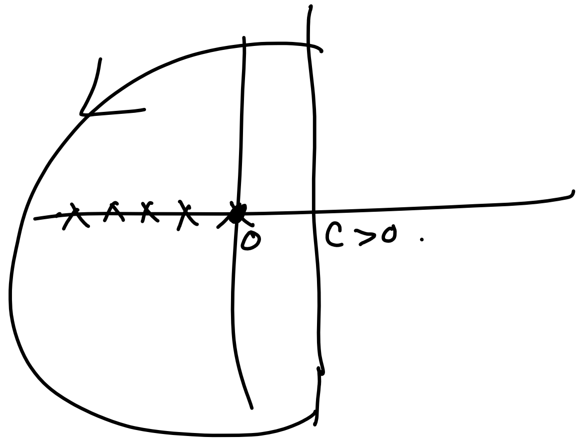
$$c_1\left[\frac{e^{\sqrt{s/k}\,a} + e^{-\sqrt{s/k}\,a}}{2}\right] = \frac{T_0}{2s}\, e^{-\sqrt{s/k}\,a}$$

$$\Rightarrow \quad c_1 = \frac{T_0}{2s}\cdot\frac{e^{-\sqrt{s/k}\,a}}{\cosh\sqrt{s/k}\,a}.$$

$$c_2 = \frac{T_0}{s} - \frac{T_0}{2s}\,\frac{e^{-\sqrt{s/k}\,a}}{\cosh\sqrt{s/k}\,a} = \frac{T_0}{s}\left[\frac{e^{\sqrt{s/k}\,a} + \cancel{e^{-\sqrt{s/k}\,a}} - \cancel{e^{-\sqrt{s/k}\,a}}}{2\cosh\sqrt{s/k}\,a}\right]$$

$$= \frac{T_0}{2s} \frac{e^{\sqrt{s/k}\,a}}{\cosh\sqrt{\frac{s}{k}}\,a}.$$

$$\Rightarrow \quad \bar{u}(x,s) = \frac{T_0}{s\cosh\sqrt{\frac{s}{k}}\,a}\left[\frac{e^{\sqrt{\frac{s}{k}}(x-a)} + e^{-\sqrt{\frac{s}{k}}(x-a)}}{2}\right]$$



$$\bar{u}(x,s) = \frac{T_0\cosh\sqrt{\frac{s}{k}}(x-a)}{s\cosh\sqrt{\frac{s}{k}}\,a}, \quad 0<x<a$$

pole: $s=0$

$$\cosh\sqrt{\frac{s}{k}}\,a = 0.$$

$$\sqrt{\frac{s}{k}}\,a = i\,\frac{(2n-1)\pi}{2}, \quad n=1,2,3,\ldots$$

Inversion gives

poles: $s_n = -\frac{(2n-1)^2\pi^2}{4}\frac{k}{a^2}, \quad n=1,2,3,\ldots$

$$u(x,t) = T_0\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos\left\{\frac{(2n-1)(a-x)\pi}{2a}\right\}\exp\left\{-(2n-1)^2\left(\frac{\pi}{2a}\right)^2 kt\right\}\right], \quad \begin{array}{l} 0<x<a \\ t>0. \end{array}$$

Solution of Linear integral equations (Volterra type)

$$u(t) = f(t) + \lambda \int_0^t k(t-\tau)\, u(\tau)\, d\tau, \quad 0<t<a$$

$$\bar{u}(s) = \bar{f}(s) + \lambda\, \bar{k}(s) \cdot \bar{u}(s).$$

$$\Rightarrow \bar{u}(s) = \frac{\bar{f}(s)}{1-\lambda \bar{k}(s)}.$$

Inversion gives

$$\Rightarrow \boxed{u(t) = \mathcal{L}^{-1}\left(\frac{\bar{f}(s)}{1-\lambda \bar{k}(s)}\right)(t)}$$

Examples:

1. Solve $$u(t) = a + \lambda \int_0^t u(\tau)\, d\tau .$$

$k(t-\tau) = 1$

L.T gives, $$\bar{u}(s) = \frac{a}{s} + \lambda \frac{1}{s} \cdot \bar{u}(s)$$

$$\Rightarrow \bar{u}(s)\left(1 - \frac{\lambda}{s}\right) = \frac{a}{s}$$

$$\Rightarrow \bar{u}(s) = \frac{a}{s-\lambda} .$$

Inversion gives $\boxed{u(t) = a e^{\lambda t}}$

2. Solve $$u(t) = a \sin t + 2 \int_0^t u'(\tau) \sin(t-\tau)\, d\tau , \quad t > 0$$

$u(0) = 0$

Soln: L.T gives

$$\bar{u}(s) = a \cdot \frac{1}{s^2+1} + 2\, \mathcal{L}\left(\frac{du}{dt}(t)\right) \cdot \frac{1}{s^2+1}$$

$$= \frac{a}{s^2+1} + 2 \cdot \frac{s\,\bar{u}(s)}{s^2+1}$$

$$\Rightarrow \quad \bar{u}(s) \cdot \left(1 - \frac{2s}{s^2+1}\right) = \frac{a}{s^2+1}$$

$$\Rightarrow \quad \bar{u}(s) = \frac{a}{(s-1)^2} \Rightarrow \boxed{u(t) = a\,t\,e^{t}.}$$

Evaluation of integrals

$$\mathcal{L} \int_a^b f(t,x)\,dx = \int_a^b \bar{f}(s,x)\,dx.$$

Examples: 1. Evaluate $I(t) = \int_0^\infty \frac{\sin tx}{x(x^2+a^2)}\,dx.$

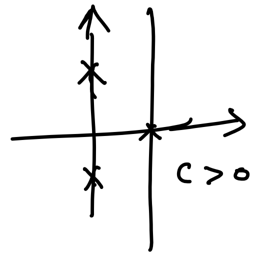
L.T gives
$$\bar{I}(s) = \int_0^\infty \frac{\cancel{x}}{s^2+x^2} \cdot \frac{1}{\cancel{x}(x^2+a^2)}\,dx$$

$$= \int_0^\infty \frac{dx}{(s^2+x^2)(x^2+a^2)}$$

$$\bar{I}(s) = \frac{1}{a^2-s^2}\int_0^\infty \left[\frac{1}{s^2+x^2} - \frac{1}{x^2+a^2}\right] dx.$$

$s^2+x^2 = 0$

$s = \sqrt{-x^2} = \pm i x$



$$= \frac{-1}{s^2-a^2}\left[\frac{1}{s}\,\frac{\pi}{2} - \frac{1}{a}\cdot\frac{\pi}{2}\right] \quad \text{since } s>0$$

$$= \frac{\pi}{2}\,\frac{1}{s^2-a^2}\,\frac{(s-a)}{sa} = \frac{\pi}{2a}\cdot\frac{1}{s(s+a)}$$

$$\bar{I}(s) = \frac{\pi}{2a^2}\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

Inversion gives

$$\Rightarrow \boxed{I(t) = \frac{\pi}{2a^2}\left(1 - e^{-at}\right)}$$

(2) Evaluate $I(t) = \int_0^\infty \frac{\sin^2 tx}{x^2}\,dx$, $t > 0$.

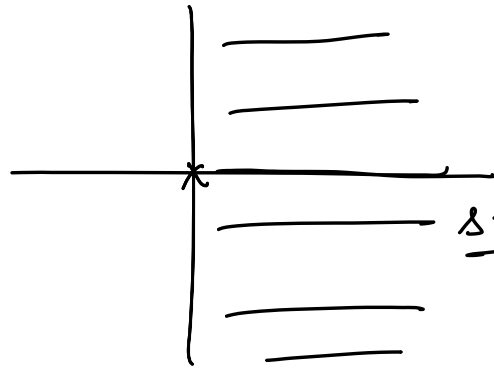
soln:

L.T gives $\bar{I}(s) = \mathcal{L}\left\{\int_0^\infty \frac{1}{x^2}\left(\frac{1-\cos 2tx}{2}\right)dx\right\}$

$$= \int_0^\infty \frac{1}{2x^2}\left(\frac{1}{s} - \frac{s}{s^2+4x^2}\right)dx$$

$$= \frac{1}{\cancel{2}}\int_0^\infty \frac{1}{\cancel{x^2}}\,\frac{\cancel{s^2}+\cancel{4x^2}^{\,2} - \cancel{s^2}}{s\,(s^2+4x^2)}\,dx$$

$$= \frac{2}{s}\int_0^\infty \frac{1}{(s^2+4x^2)}\,dx.$$



$s > 0$

$$= \frac{1}{s} \int_0^\infty \frac{dy}{(y^2+s^2)}$$

$2x = y$

$2dx = dy$

$$= \frac{1}{s} \frac{1}{s} \cdot \left(\tan^{-1} \frac{y}{s}\right)\Big|_0^\infty$$

$$\bar{I}(s) = \frac{1}{s^2} \cdot \frac{\pi}{2} \quad \text{since } \underline{s > 0} .$$

Inverse transform now gives

$$\boxed{I(t) = \frac{\pi}{2} t, \quad t > 0}$$

3. Evaluate $I(t) = \int_0^\infty \frac{x \sin xt}{x^2 + a^2}\, dx, \quad t > 0.$

$$\frac{1}{x^2+s^2} - \frac{a^2}{s^2-a^2}\left(\frac{1}{x^2+a^2} - \frac{1}{x^2+s^2}\right)$$

$$\frac{x^2+a^2}{(x^2+s^2)(x^2+a^2)} - \frac{a^2}{(x^2+a^2)(x^2+s^2)}$$

L.T give $\quad \bar{I}(s) = \int_0^\infty \frac{x}{x^2+a^2} \cdot \frac{x}{s^2+x^2}\, dx.$

$$= \int_0^\infty \frac{(x^2+a^2-a^2)}{(x^2+a^2)(s^2+x^2)}\, dx.$$

$$= \int_0^\infty \frac{\cancel{(x^2+a^2)}}{\cancel{(x^2+a^2)}(s^2+x^2)}\, dx - a^2 \int_0^\infty \frac{dx}{(x^2+a^2)(s^2+x^2)}$$

$$= \int_0^\infty \frac{dx}{(x^2+s^2)} - \frac{a^2}{(s^2-a^2)} \int_0^\infty \left(\frac{1}{x^2+a^2} - \frac{1}{x^2+s^2}\right) dx.$$

$$= \frac{1}{s}\cdot\frac{\pi}{2} - \frac{a^2}{(s^2-a^2)}\left(\frac{1}{a}\cdot\frac{\pi}{2} - \frac{1}{s}\frac{\pi}{2}\right),$$ since $a>0$,

$$= \frac{\pi}{2s} - \frac{\cancel{a^2}}{\cancel{s^2-a^2}}\cdot\frac{\pi}{2}\,\frac{\cancel{(s-a)}}{s\cancel{a}} = \frac{\pi}{2}\left(\frac{1}{s} - \frac{a}{(s+a)s}\right)$$

$$\bar{I}(s) = \frac{\pi}{2s}\left(1 - \frac{a}{s+a}\right) = \frac{\pi}{2\cancel{s}}\cdot\frac{\cancel{s}}{s+a} = \underline{\frac{\pi}{2}\cdot\frac{1}{s+a}}.$$

Inversion gives $\boxed{I(t) = \frac{\pi}{2}\, e^{-at}, \quad t>0,\ a>0}$ ✓

if $\underline{a<0}$. ✓

$$\frac{\pi}{2s} + \frac{\pi}{2}\frac{a^2}{s^2-a^2}\quad \frac{s+a}{sa}$$

$$\frac{\pi}{2s} + \frac{\pi}{2s}\left(\frac{a}{s-a}\right)$$

$$= \frac{\pi}{2s}\left(1 + \frac{a}{s-a}\right) = \frac{\pi}{2\cancel{s}}\cdot\frac{\cancel{s}}{s-a}$$

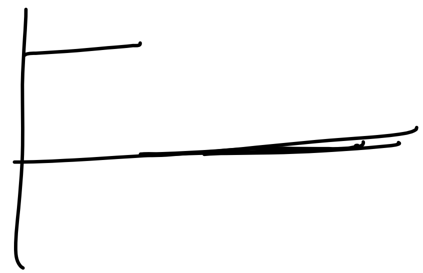
$$= \frac{\pi}{2}\cdot\frac{1}{s-a}$$

$$\underline{I(t) = \frac{\pi}{2}\cdot e^{at}}, \text{ if } a<0,\ \underline{t>0}$$ ✓

ODE with piecewise continuous forcing :

I.V.P $\begin{cases} y'' + y = g(t) \\ y(0) = 0 = y'(0) \end{cases}$

$$g(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t \geq 1 \end{cases}$$



$$g(t) = 1 - H(t-1)$$

$$\mathcal{L}\big(H(t)\big)(s) = \frac{1}{s}.$$

$$\bar{g}(s) = \frac{1}{s} - e^{-s}\frac{1}{s}.$$

$$\bar{g}(s) = \frac{1-e^{-s}}{s}$$

$$s^2\bar{y}(s) - \cancel{s\,y(0)} - \cancel{y'(0)} + \bar{y}(s) = \frac{1-e^{-s}}{s}$$

$$\Rightarrow \bar{y}(s)(1+s^2) = \frac{1-e^{-s}}{s} \Rightarrow \bar{y}(s) = \frac{1-e^{-s}}{s(1+s^2)}$$

$$\bar{y}(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right)\cdot\left(1 - e^{-s}\right) = \underline{\frac{1}{s}} - \underline{\frac{e^{-s}}{s}} - \underline{\frac{s}{s^2+1}} + \frac{s}{s^2+1}e^{-s}$$

Inversion gives

$$y(t) = 1 - H(t-1) - \cos t + \cos(t-1)\,H(t-1)$$

$$\boxed{y(t) = 1 - \cos t - H(t-1)\left(1 - \cos(t-1)\right),\ t > 0} \checkmark$$

* Solve $y'' + \underline{4y'} + 4y = g(t)$, where $g(t) = \begin{cases} t, & 1 \le t < 3 \\ 0, & 0 < t < 1,\ t \geq 3. \end{cases}$

$y(0) = 0,\ y'(0) = 1$

Soln: Since $g(t) = t\left(H(t-1) - H(t-3)\right)$

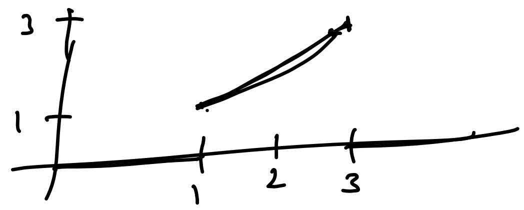
$$g(t) = (t-1)\,H(t-1) + 1\cdot H(t-1) - (t-3)\,H(t-3) - 3\,H(t-3)$$

$$\bar{g}(s) = \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-3s}}{s^2} - 3\frac{e^{-3s}}{s}.$$

$$\bar{g}(s) = \frac{(s+1)\,e^{-s}}{s^2} - e^{-3s}\frac{(3s+1)}{s^2}$$

$$s^2\bar{y}(s) - 1 + 4s\bar{y}(s) + 4\bar{y}(s) = \frac{(s+1)}{s^2}e^{-s} - e^{-3s}\cdot\frac{(3s+1)}{s^2}$$

$$\bar{y}(s)\left(s^2+4s+4\right) = 1 + \frac{(s+1)}{s^2}e^{-s} - \frac{(3s+1)}{s^2}e^{-3s}$$



$t = t\left(H(t-1) - H(t-3)\right)$ $\quad 1<t<3$

$0 =$ $\quad 0<t<1$

$0 =$ $\quad t \geq 3$

$$\Rightarrow \quad \bar{y}(s) = \frac{1}{(s+2)^2} + \frac{s+1}{s^2(s+2)^2}e^{-s} - \frac{(3s+1)}{s^2(s+2)^2}e^{-3s}.$$

$$\bar{y}(s) = \frac{1}{(s+2)^2} + \frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{(s+2)^2}\right)e^{-s} - \frac{1}{4}\left[\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s+2} - \frac{5}{(s+2)^2}\right]e^{-3s}.$$

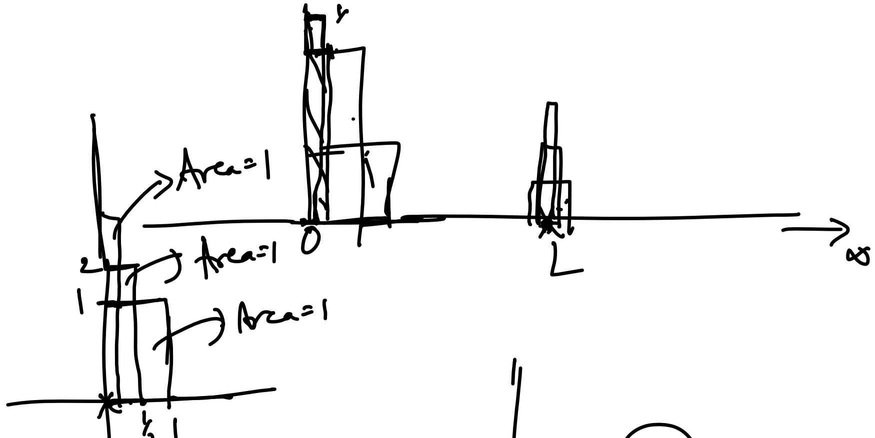
Inversion gives

$$y(t) = t\,e^{-2t} + \frac{1}{4}\left((t-1)H(t-1) - (t-1)H(t-1)\,e^{-2(t-1)}\right)$$

$$-\frac{1}{4}\left[2H(t-3) + (t-3)H(t-3) - 2\cdot e^{-2(t-3)}H(t-3) - 5\cdot(t-3)\,e^{-2(t-3)}H(t-3)\right], \quad t>0$$

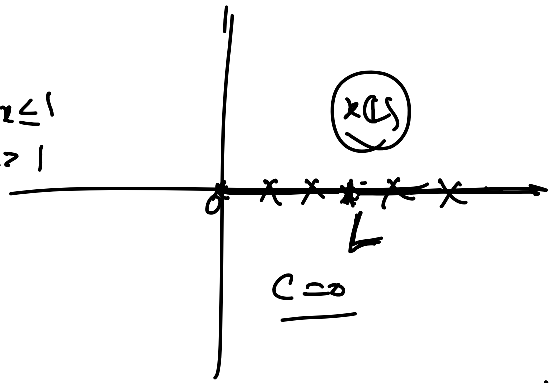
$$\underline{\delta(x)} \longrightarrow \underline{\int_{-\infty}^{\infty} f(x)\,dx =}$$

$$\int_{-\infty}^{\infty} \delta(x-L)\,dx = 1, \qquad \underline{L>0}$$



$$\mathcal{L}\left(\delta(x-L)\right)(s) = \int_0^{\infty} \delta(x-L)\, e^{-sx}\,dx$$

$$f_1(x) = \begin{cases} 1, & 0<x\le 1 \\ 0, & x>1 \end{cases}$$



$$= e^{-sL}, \qquad \delta(x) = \lim_{n\to\infty} \underline{f_n(x)}$$

$$f_n(x) = \begin{cases} n, & 0<x<\frac{1}{n} \\ 0, & x>\frac{1}{n} \end{cases}$$

$$\mathcal{L}\left(\delta(x)\right) = \int_0^{\infty} e^{-sx}\,\delta(x)\,dx = \lim_{n\to\infty}\int_0^{\infty} e^{-sx} f_n(x)\,dx = 1 \checkmark$$

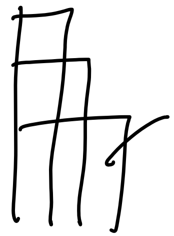
$$x(t) = x(L)\cdot\delta(t-L) \checkmark$$

As $L \to 0$ $\quad \mathcal{L}(\delta(x-L))(s) = e^{-sL}$

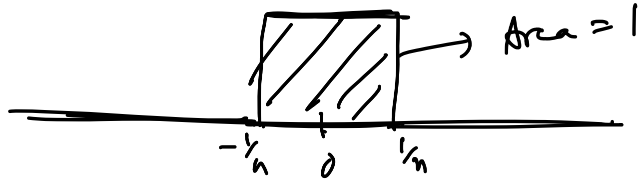
$$\downarrow$$

$$\mathcal{L}(\delta(x))(s) = 1 \quad ✓$$

$$\int_0^\infty \delta(x)\,dx = 1 \quad ✓$$



$$\delta(x) \overset{\text{weak}}{:=} \lim_{n\to\infty} f_n(x) \quad ✓$$

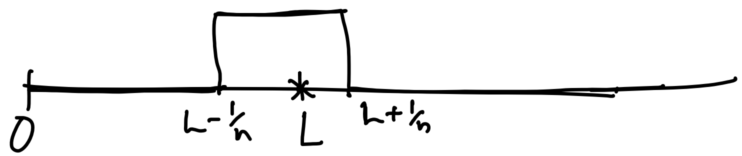


$$f(0) = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, f(x) = \int_{-\infty}^{\infty} \delta(x)\, f(x)\,dx,$$

for continuous function $f(x)$

$$\delta(t-L) := \lim_{n\to\infty} f_n(t)$$

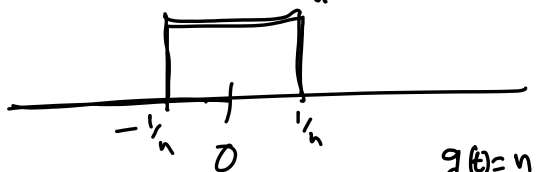
$f_n(t) = \frac{n}{2}, \quad n = 1, 2, 3, \dots$



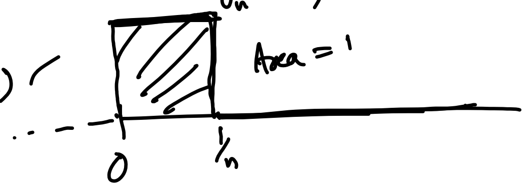
$$\mathcal{L}(\delta(t-L))(s) = \int_0^\infty \delta(t-L)\, e^{-st}\, dt$$

$$= \int_{-\infty}^\infty \delta(t-L)\, e^{-st}\, dt = e^{-sL}.$$

$f_n(t) = \frac{2}{n}, \quad n = 1, 2, 3, \dots$



$g_n(t) = n, \quad n = 1, 2, 3, \dots$



$$\int_0^\infty f(t)\, e^{-st}\, dt = \lim_{n\to\infty} \int_0^\infty g_n(t)\, e^{-st}\, dt \iff \delta(t) \overset{\text{weak}}{:=} \lim_{n\to\infty} g_n(t),$$

$$\int_0^\infty \delta(t)\, dt = 1$$

Taking $L \to 0$, $\mathcal{L}(\delta(t))(s) = 1.$

$$\int_0^\infty \delta(t)\, e^{-st}\, dt = \frac{1}{2}.$$

$$\mathcal{L}(f(t))(s) = \int_0^\infty \delta(t)\, e^{-st}\, dt$$

$$= \lim_{n\to\infty} \int_0^\infty g_n(t)\, e^{-st}\, dt$$

$$= \lim_{n\to\infty} \int_0^{1/n} n\, e^{-st}\, dt$$

$$= \lim_{n\to\infty} n \cdot \left. \frac{e^{-st}}{-s} \right|_0^{1/n}$$

$$= \lim_{n\to\infty} \frac{n}{s} \left(1 - e^{-s/n}\right)$$

$$= \lim_{n\to\infty} \frac{\left(1 - e^{-s/n}\right)}{s/n}.$$

$$= \lim_{s/n \to 0} \frac{1 - e^{-s/n}}{s/n}$$

$$= \lim_{x\to 0} \frac{1 - e^{-x}}{x}$$

$$\boxed{\mathcal{L}(\delta(t))(s) = \lim_{x\to 0} e^{-x} = 1}$$

* Solve $y'' + \pi^2 y = \delta(t-1)$, $t > 0$

$y(0) = 1$, $y'(0) = 0$.

Sol:
$$s^2 \bar{y}(s) - s + \pi^2 \bar{y}(s) = e^{-s} \quad \text{by Laplace transform.}$$

$$\bar{y}(s)\left(s^2+\pi^2\right) = s + e^{-s}$$

$$\bar{y}(s) = \frac{s+e^{-s}}{s^2+\pi^2} = \frac{s}{s^2+\pi^2} + \frac{1}{\pi}\,\frac{\pi}{s^2+\pi^2}\,e^{-s}.$$

Inversion gives $\boxed{y(t) = \cos\pi t + \frac{1}{\pi} H(t-1)\left(\sin\pi(t-1)\right), \quad t>0}$ ✓

* Solve $y'' - 4y' + 3y = (2t+1)\,\delta(t-\tfrac{1}{2}), \quad t>0$

$y(0)=0, \quad y'(0)=2.$

Soln: Application of L.T gives

$$s^2 \bar{y}(s) - 2 - 4 s \bar{y}(s) + 3 \bar{y}(s) = \int_0^\infty (2t+1)\, \delta(t-\tfrac{1}{2})\, e^{-st}\, dt$$

$$\Rightarrow \quad \bar{y}(s)\left(s^2-4s+3\right) = 2 + 2e^{-\frac{s}{2}}.$$

$$\Rightarrow \quad \bar{y}(s) = \frac{2}{(s-3)(s-1)} + \frac{2e^{-s/2}}{(s-3)(s-1)}$$

$$\bar{y}(s) = \frac{1}{s-3} - \frac{1}{s-1} + \left(\frac{1}{s-3} - \frac{1}{s-1}\right) e^{-s/2}.$$

Laplace inversion gives $\boxed{y(t) = e^{3t} - e^{t} + e^{3(t-\frac{1}{2})} H(t-\tfrac{1}{2}) + e^{(t-\frac{1}{2})} H(t-\tfrac{1}{2}), \quad t>0}$ ✓

* $y'' + y' = \delta(t-1) - \delta(t-2)$,

$y(0) = 0, \quad y'(0) = 0.$

Soln:

$$s^2 \bar{y}(s) + s\,\bar{y}(s) = e^{-s} - e^{-2s}$$

$$\Rightarrow \quad \bar{y}(s) = \frac{1}{s(s+1)}\left(e^{-s} - e^{-2s}\right)$$

$$\bar{y}(s) = \left(\frac{1}{s} - \frac{1}{s+1}\right)\left(e^{-s} - e^{-2s}\right)$$

Inversion gives $\boxed{y(t) = H(t-1) - H(t-2) - e^{-(t-1)} H(t-1) + e^{-(t-2)} H(t-2), \quad t > 0}$ ✓