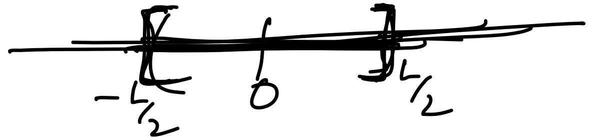
Fourier integral Theorem:

Let $f(x)$ be piecewise smooth function in every finite interval in $(-\infty, \infty)$

and $f(x)$ be absolutely integrable in $(-\infty, \infty)$.

$\left(\text{ie} \int_{-\infty}^{\infty} |f(x)|\, dx < \infty \right)$.



$f(x)$ or $\frac{f(x^+)+f(x^-)}{2}$ = F. Series

As $L \to \infty$,

Then

$$f(x) \text{ or } \frac{f(x^+)+f(x^-)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{i\xi(x-t)}\, dt\, d\xi .$$

or

$$= \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos\left(\xi(x-t)\right) dt\, d\xi$$

✓ Intuitive proof:

Since $f(x)$ or $\frac{f(x^+)+f(x^-)}{2} = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}$, $\omega_0 = \frac{2\pi}{L}$.

$x \in \left(-\frac{L}{2}, \frac{L}{2}\right)$

for any $L>0$.

where $c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-in\omega_0 x}\, dx$ ✓.

$$\Rightarrow \quad f(x) \text{ or } \frac{f(x^+)+f(x^-)}{2} = \frac{1}{L}\sum_{n=-\infty}^{\infty}\int_{-L/2}^{L/2} f(t)\, e^{-in\omega_0 t}\, dt\, e^{in\omega_0 x}$$

$$\overset{?}{=} \frac{1}{L}\int_{-L/2}^{L/2}\sum_{n=-\infty}^{\infty} f(t)\, e^{in\frac{2\pi}{L}(x-t)}\, dt$$

Let $s_n = \frac{2n\pi}{L}$, $s_{n+1}-s_n = \Delta s_n = \frac{2\pi}{L} = \omega_0$.

$$= \frac{1}{2\pi} \int_{-L/2}^{L/2} \sum_{n=-\infty}^{\infty} f(t)\, e^{i s_n (x-t)} \Delta s_n \, dt .$$

$$= \frac{1}{2\pi} \int_{-L/2}^{L/2} \lim_{k \to \infty} \underline{\sum_{n=-k}^{k} f(t)\, e^{i s_n (x-t)} \Delta s_n} \, dt$$

As $L \to \infty$, $\underline{\Delta s_n \to 0}$.

$$= \frac{1}{2\pi} \underline{\lim_{L \to \infty}} \int_{-L/2}^{L/2} \underline{\lim_{k \to \infty} \sum_{n=-k}^{k} f(t)\, e^{i s_n (x-t)} \Delta s_n} \, dt$$

$$\underline{f(x) \text{ or } \frac{f(x^+)+f(x^-)}{2}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{i \xi (x-t)} \, \underline{d\xi} \, \underline{dt} \checkmark$$

Let $\xi = -\xi$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{-i \xi (x-t)} \, d\xi \, dt \checkmark$$

Fourier transform:

$$\mathcal{F}(f(x))(\xi) = \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\xi t}\, dt \quad \text{or} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\xi t}\, dt$$

Inverse Fourier transform:

$$f(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{i\xi x}\, d\xi \quad \text{or} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{-i\xi x}\, d\xi$$

(from Fourier integral theorem).

Remark:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{i\xi(x-t)}\, dt\, d\xi = \frac{1}{2\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{i\xi(x-t)}\, dt\, d\xi + \frac{1}{2\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{-i\xi(x-t)}\, dt\, d\xi$$

$$= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\left[\underline{e^{i\xi(x-t)} + e^{-i\xi(x-t)}}\right] dt\, d\xi$$

$f(x)$ or $\underline{\dfrac{f(x^+)+f(x^-)}{2}}$ $$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\, \underline{\cos(\xi(x-t))}\, dt\, d\xi$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\left(\cos \xi x \cos \xi t + \underline{\sin \xi x \sin \xi t}\right) dt\, d\xi \checkmark$$

$$\int_0^\infty \sin \xi x \int_{-\infty}^{\infty} \cancel{\underline{f(t) \sin \xi t}}\, dt\, d\xi = \underline{0}$$

$$\int_0^\infty \cos \xi x \int_{-\infty}^{\infty} \cancel{\underline{f(t) \cos \xi t}}\, dt\, d\xi = \underline{0}$$

<u>If $f(x)$ is even function</u> on $(-\infty,\infty)$ i.e, $f(-x) = f(x)$, $\forall x$; then

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} \underline{f(t) \cos \xi x \cos \xi t}\, dt\, d\xi .$$

$$= \frac{2}{\pi}\int_0^\infty \int_0^\infty \underline{f(t) \cos \xi t\, dt}\, \cos \xi x\, d\xi \checkmark$$

Fourier Cosine transform:

$$F_c(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \xi t \, dt \checkmark$$

Inverse Fourier Cosine transform:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\xi) \cos \xi x \, d\xi$$

If $f(x)$ is odd function in $(-\infty, \infty)$

or

$f(x), x \in (0, \infty) \longrightarrow$ extend as odd function over $(-\infty, \infty)$

then $f(x) = \frac{1}{\pi} \int_0^\infty \underline{\int_{-\infty}^\infty \underline{f(t) \sin \xi t} \, dt} \sin \xi x \, d\xi.$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{\sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \xi t \, dt}_{} \sin \xi x \, d\xi$$

Fourier Sine transform:

$$F_s(\xi) := \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \xi t \, dt$$

Inverse Fourier sine transform:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\xi) \sin \xi x \, d\xi$$

Example: If $f(x) = \begin{cases} 0; & x<0 \\ \frac{1}{2} & x=0 \\ e^{-x}, & x>0 \end{cases}$, Then $\int_{-\infty}^{\infty} |f(x)|\, dx = \int_{0}^{\infty} e^{-x}\, dx < \infty$

$f(x)$ is picewise smooth function on every finite interval and absolutely integrable over $(-\infty, \infty)$

and by Fourier integral Theorem,

$$f(x) \text{ or } \frac{f(x^+)+f(x^-)}{2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\xi t}\, dt \right) e^{i\xi x}\, d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t(1+i\xi)}\, dt \right) e^{i\xi x}\, d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{e^{i\xi x}}{1+i\xi}\, d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos \xi x + i \sin \xi x}{1+i\xi}\, d\xi$$

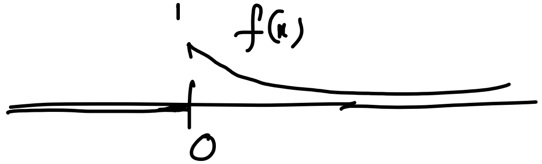
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(1-i\xi)(\cos \xi x + i \sin \xi x)}{1+\xi^2}\, d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos \xi x + \xi \sin \xi x}{1+\xi^2}\, d\xi + \frac{i}{2\pi} \cancel{\int_{-\infty}^{\infty} \left(\frac{-\xi \cos \xi x + \sin \xi x}{1+\xi^2} \right) d\xi}$$

$$f(x) \text{ or } \frac{f(x^+)+f(x^-)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos \xi x + \xi \sin \xi x}{1+\xi^2}\, d\xi \quad ✓$$

$$\frac{1+0}{2} = \frac{1}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\xi}{1+\xi^2} = \frac{1}{2\pi}\cdot \tan^{-1}\xi \Big|_{-\infty}^{\infty}$$

$$f(x) = \begin{cases} 0, & x<0 \\ \frac{1}{2}, & x=0 \\ e^{-x}, & x>0 \end{cases}$$



$$= \frac{1}{2\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{2\cancel{\pi}}\cdot\cancel{\pi} = \frac{1}{2} \checkmark$$

$$\Rightarrow \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos \xi x + \xi \sin \xi x}{1+\xi^2}\, d\xi, \quad \forall\, x \in (-\infty, \infty)$$

Example: Show that $e^{-x}\cos x = \frac{2}{\pi}\int_0^{\infty} \frac{(\xi^2+2)\cos x\xi}{\xi^4+4}\, d\xi;\ x>0$ ✓

Use Fourier integral Theorem,

$$\int_0^\infty e^{-t}\cos xt\,dt = \frac{1}{1+x^2}$$

$$e^{-x}\cos x = \sqrt{\frac{2}{\pi}}\int_0^\infty \left(\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-t}\cos t\cos\xi t\,dt\right)\cos\xi x\,d\xi,\ x>0.$$

$$= \frac{1}{\pi}\int_0^\infty \int_0^\infty e^{-t}\left(\cos(1+\xi)t + \cos(1-\xi)t\right)dt\,\cos\xi x\,d\xi,\ x>0$$

$$= \frac{1}{\pi}\int_0^\infty \left[\frac{1}{1+(1+\xi)^2} + \frac{1}{1+(1-\xi)^2}\right]\cos\xi x\,d\xi,\ x>0.$$

$$= \frac{1}{\pi}\int_0^\infty \left(\frac{1+(1-\xi)^2 + 1+(1+\xi)^2}{\left(1+(1+\xi)^2\right)\left(1+(1-\xi)^2\right)}\right)\cos\xi x\,d\xi,\ x>0$$

$$= \frac{1}{\pi}\int_0^\infty \frac{(4+2\xi^2)\cos\xi x\,d\xi}{(2+\xi^2+2\xi)(2+\xi^2-2\xi)} = \frac{1}{\pi}\int_0^\infty \frac{(\xi^2+2)\,2\cos\xi x\,d\xi}{(2+\xi^2)^2-4\xi^2} = \frac{2}{\pi}\int_0^\infty \frac{(\xi^2+2)\cos\xi x\,d\xi}{\xi^4+4},\ x>0.$$

Fourier transform of $\delta(x)$:

$*\quad \hat{\delta}(\xi) = \frac{1}{\sqrt{2\pi}}$ ✓

Defn: (weak limit) $\lim_{\epsilon \to 0} f_\epsilon(x) = \delta(x)$ if

generalized function.

$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} f_\epsilon(x)\, g(x)\, dx = \int_{-\infty}^{\infty} \delta(x) g(x)\, dx = g(0)$ ✓

usual limit $\quad \forall\, g \in C_c^\infty(\mathbb{R})$ ✓

$\delta(x) := \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} = \lim_{\epsilon \to 0} f_\epsilon(x)$, where $f_\epsilon(x) = \begin{cases} 1/\epsilon, & x \in [-\epsilon/2, \epsilon/2] \\ 0, & \text{otherwise}. \end{cases}$

$$\hat{\delta}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} f_\epsilon(x)\, e^{-i\xi x}\, dx \quad ✓$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f_\epsilon(x)\, e^{-i\xi x}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{-\epsilon/2}^{\epsilon/2} e^{-i\xi x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \cdot \frac{e^{-i\xi x}}{-i\xi} \Bigg|_{-\epsilon/2}^{\epsilon/2} .$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \cdot \frac{i}{\xi} \left[e^{-i\xi \frac{\epsilon}{2}} - e^{i\xi \frac{\epsilon}{2}} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \lim_{\epsilon \to 0} \cdot \frac{1}{\epsilon} \frac{i}{\xi} (-2i) \sin \frac{\xi\epsilon}{2} .$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \to 0} \frac{2}{\epsilon\xi} \cdot \sin\left(\frac{\xi\epsilon}{2}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\frac{\xi\epsilon}{2} \to 0} \frac{\sin\left(\frac{\xi\epsilon}{2}\right)}{\left(\xi\epsilon/2\right)} = \frac{1}{\sqrt{2\pi}}$$

Heavyside function:

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

$$\delta(x) = \frac{d}{dx}\big(H(x)\big).$$

$$\hat{\delta}(\xi) = \frac{1}{\sqrt{2\pi}} = \widehat{H'}(\xi) = i\xi\,\hat{H}(\xi)$$

$$\hat{H}(\xi) = \frac{-i}{\xi\sqrt{2\pi}}$$

$$\hat{f'}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)\, e^{-i\xi x}\, dx$$

$$= \cancel{\frac{1}{\sqrt{2\pi}} \cdot f(x)\, e^{-i\xi x}\Big|_{-\infty}^{\infty}} + i\xi \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx$$

$$= i\xi\, \hat{f}(\xi)$$

$$\delta(x) = \frac{d}{dx}\left(H(x)\right).$$

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

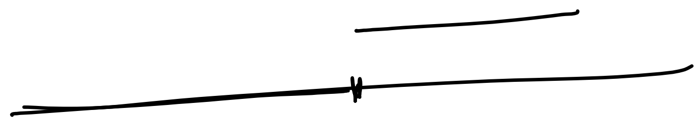
$$\frac{d}{dx}\left(H(x)\right) = \lim_{\Delta x \to 0} \frac{H(x+\Delta x) - H(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1 - 1}{\Delta x} = 0, \text{ if } x > 0$$

$$\frac{d}{dx}\left(H(x)\right) = \lim_{\Delta x \to 0} \frac{H(x+\Delta x) - H(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0 \quad \text{if } x < 0$$

$$\delta(x) = \frac{d}{dx}(H(x)) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$



$$\left.\frac{d}{dx}(H(x))\right|_{x=0} = \lim_{\Delta x \to 0} \frac{H(\Delta x) - H(0)}{\Delta x}$$

$$\left.\begin{aligned} &= \lim_{\Delta x \to 0^+} \frac{1-0}{\Delta x} = \underline{\underline{\infty}} \\ &= \lim_{\Delta x \to 0^-} \frac{0-0}{\Delta x} = 0 \end{aligned}\right\}$$

weak equality

$$\Rightarrow \quad \underline{\underline{\delta(x) = H'(x)}} \iff \int_{-\infty}^{\infty} H'(x)\, g(x)\, dx = \int f(x)\, g(x)\, dx \checkmark$$

$$\forall g \in \underline{C}_c^{\infty}(\mathbb{R}) \checkmark$$

$$\hat{\delta}(\xi) = \frac{1}{\sqrt{2\pi}} = \widehat{\left(\frac{d}{dx}H(x)\right)}(\xi) = \cancel{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d}{dx}(H(x))\,e^{-i\xi x}\,dx}$$

$$\cancel{= \frac{H(x)\,e^{-i\xi x}}{\sqrt{2\pi}}\Big|_{-\infty}^{\infty} + \frac{i\xi}{\sqrt{2\pi}}\int_{-\infty}^{\infty}H(x)\,e^{-i\xi x}\,dx}$$

$$= \cancel{\frac{e^{-i\xi\infty}}{\sqrt{2\pi}}} + \frac{i\xi}{\sqrt{2\pi}}\hat{H}(\xi)$$

$$\widehat{\frac{df}{dx}}(\xi) = i\xi\,\hat{f}(\xi).$$

if $f(\pm\infty) = 0$

$$f(x) \overset{\text{strong}}{=} g(x)$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\cdot h(x)\,dx = \int_{-\infty}^{\infty} g(x)\,h(x)\,dx, \quad \forall h \in C_c^{\infty}(\mathbb{R})$$

$$\Rightarrow f(x) \overset{\text{weak}}{=} g(x)$$

If I want $\hat{H}(\xi)$?

$$H(x) \overset{\text{strong}}{=} \lim_{\alpha\to 0^+} H_\alpha(x), \text{ where } H_\alpha(x) = \begin{cases} 0, & x \le 0 \\ e^{-\alpha x}, & x > 0 \end{cases}$$

$$\hat{H}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(x)\,e^{-i\xi x}\,dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{\alpha \to 0^+} H_\alpha(x)\, e^{-i\xi x}\, dx$$

$$\overset{?}{=} \frac{1}{\sqrt{2\pi}} \lim_{\alpha \to 0^+} \int_{-\infty}^{\infty} H_\alpha(x)\, e^{-i\xi x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\alpha \to 0^+} \int_{0}^{\infty} e^{-\alpha x}\, e^{-i\xi x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\alpha \to 0^+} \left. \frac{e^{-(\alpha + i\xi)x}}{-(\alpha + i\xi)} \right|_{0}^{\infty}$$

$$\underline{\hat{H}(\xi)} = \frac{1}{\sqrt{2\pi}} \lim_{\alpha \to 0^+} \frac{1}{\alpha + i\xi} = \frac{1}{i\xi\sqrt{2\pi}}$$

$$\mathcal{F}^{-1}\left(\frac{1}{i\xi\sqrt{2\pi}}\right)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{i\xi\sqrt{2\pi}}e^{i\xi x}\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\cos\xi x + i\sin\xi x}{i\xi}\,d\xi$$

$$= \frac{\cancel{2}}{\cancel{2}\pi}\int_{0}^{\infty}\frac{\sin\xi x}{\xi}\,d\xi \checkmark$$

$$= \begin{cases}\frac{1}{\pi}\cdot\frac{\pi}{2} & x>0\\ \frac{-1}{\pi}\cdot\frac{\pi}{2}, & x<0\end{cases}$$

$$= \begin{cases}\frac{1}{2}, & x>0\\ -\frac{1}{2}, & x<0\end{cases} \neq H(x) \checkmark$$

$$\int_0^\infty \frac{\sin\xi x}{\xi}\,d\xi = \begin{cases}\pi/2, & \text{if } x>0\\ -\pi/2 & \text{if } x<0\end{cases}$$

$$\mathcal{F}\mathcal{F}^{-1}\left(\frac{1}{i\xi\sqrt{2\pi}}\right)(x) = \mathcal{F}\left(H(x) - \frac{1}{2}\right) ✓$$

$$\Rightarrow \frac{1}{i\xi\sqrt{2\pi}} = \hat{H}(\xi) - \mathcal{F}\left(\frac{1}{2}\right). ✓$$

$$\Rightarrow \hat{H}(\xi) = \mathcal{F}\left(\frac{1}{2}\right) + \frac{1}{i\xi\sqrt{2\pi}} ✓$$

$$\mathcal{F}^{-1}\left(\delta(\xi)\right)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(\xi)\, e^{i\xi x}\, d\xi$$

$$= \frac{1}{\sqrt{2\pi}} ✓$$

$$\Rightarrow \mathcal{F}\left(\frac{1}{\sqrt{2\pi}}\right) = \delta(\xi) ✓ \Rightarrow \mathcal{F}(1) = \sqrt{2\pi}\cdot\delta(\xi)$$

Linearity:

$$\widehat{f_1 + cf_2}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f_1 + cf_2)(x)\, e^{-i\xi x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(x)\, e^{-i\xi x}\, dx + \frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(x)\, e^{-i\xi x}\, dx$$

$$= \hat{f}_1(\xi) + c\hat{f}_2(\xi) ✓$$

$$\mathcal{F}(1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi x}\, dx$$

$$\mathcal{F}(\delta(x))(\xi) = \hat{\delta}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)\, e^{-i\xi x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} ✓$$

$$\delta(x) \longleftrightarrow \frac{1}{\sqrt{2\pi}}.$$

$$\delta(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\, e^{i\xi x}\, d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\, d\xi$$

$$\Rightarrow \hat{H}(\xi) = \frac{1}{2}\cdot\sqrt{2\pi}\;\delta(\xi) + \frac{1}{i\xi\sqrt{2\pi}}.$$

$$\boxed{\hat{H}(\xi) = \sqrt{\frac{\pi}{2}}\;\delta(\xi) + \frac{1}{i\sqrt{2\pi}\;\xi}.}$$ ✓

We evaluate $\displaystyle\int_0^\infty \frac{\sin \xi x}{x}\,dx$, for $\xi > 0$ or $\xi < 0$.

Fourier Sine transform of $\underline{\underline{e^{-ax}}}$, $\underline{a>0}$. $x>0$. $\underline{a \in (0,\infty)}$

$$F_s(e^{-ax})(\xi) = \sqrt{\frac{2}{\pi}}\int_0^\infty \underline{e^{-ax}\sin \xi x\,dx}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\xi}{\xi^2 + a^2}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ax} \sin \xi x \, dx = \frac{1}{\sqrt{2\pi}} \frac{\xi}{\xi^2 + a^2}.$$

Differentiate w.r.to 'a' on both sides, we get

$$\frac{1}{\sqrt{2\pi}} \cancel{-} \int_0^\infty x e^{-ax} \sin \xi x \, dx = \frac{1}{\sqrt{2\pi}} \frac{\cancel{-} \xi \cdot 2a}{(\xi^2 + a^2)^2}$$

$$\Rightarrow F_s(x e^{-ax}) = \frac{2\xi a}{\sqrt{2\pi} (\xi^2 + a^2)^2}.$$

Integrate the above equality w.r.to 'a' from a to ∞; we get

$$\underbrace{\int_0^\infty e^{-ax} \sin \xi x \, dx}_{I} = \cancel{\frac{e^{-ax}}{-a} \sin \xi x \Big|_0^\infty} + \frac{\xi}{a} \int_0^\infty e^{-ax} \cos \xi x \, dx$$

$$= \frac{\xi}{a} \left[-\frac{e^{-ax}}{a} \cos \xi x \Big|_0^\infty - \frac{\xi}{a} \underbrace{\int_0^\infty e^{-ax} \sin \xi x \, dx}_{I} \right]$$

$$I = \frac{\xi}{a^2} - \frac{\xi^2}{a^2} I$$

$$\Rightarrow \left(\frac{a^2 + \xi^2}{\cancel{a^2}} \right) I = \frac{\xi}{\cancel{a^2}}$$

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty \left[-\frac{e^{-ax}}{x}\right]_{a=a}^{a=\infty} \sin \xi x \, dx = \frac{\xi}{\sqrt{2\pi}} \cdot \int_a^\infty \frac{1}{\xi^2+a^2}\, da \,.$$

$$\underline{\cancel{\frac{1}{\sqrt{2\pi}}}\int_0^\infty \left(\frac{e^{-ax}}{x}\right)\sin \xi x \, dx} = \cancel{\frac{1}{\sqrt{2\pi}}}\int_a^\infty \frac{d(a/\xi)}{\underline{1+(a/\xi)^2}} \,.$$

$\left(\frac{\infty}{\xi}\right) = -\infty$ if $\underline{\xi < 0}$

$$a/\xi = t \Rightarrow d(a/\xi) = dt$$

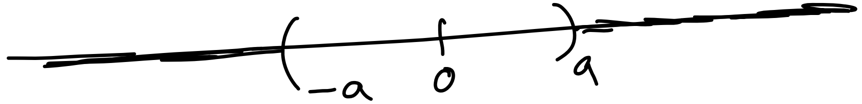
$$= \cancel{\frac{1}{\sqrt{2\pi}}}\int_{a/\xi}^\infty \frac{dt}{1+t^2} = \cancel{\frac{1}{\sqrt{2\pi}}} \tan^{-1} t \Big|_{a/\xi}^{\pm\infty} \,, \quad \text{if } \xi \gtrless 0$$

$$= \cancel{\frac{1}{\sqrt{2\pi}}}\left[\pm\frac{\pi}{2} - \tan^{-1}\left(\frac{a}{\xi}\right)\right]$$

As $a \to 0^+$, $\displaystyle \int_0^\infty \frac{\sin \xi x}{x}\, dx = \begin{cases} \pi/2 \,, & \text{if } \xi > 0 \\ -\pi/2 \,, & \text{if } \xi < 0 \end{cases}$ ✓

problem: Find the Fourier transform of

$$f(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$$



and deduce the value of $\int_0^\infty \frac{\sin ax}{x}\,dx$; $a > 0$

Solution:

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-i\xi x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \left(\cos \xi x - i \sin \xi x\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos \xi x \,dx = \frac{1}{\sqrt{2\pi}} \frac{\sin \xi x}{\xi} \Bigg|_{-a}^{a} = \frac{\sin \xi a + \sin \xi a}{\sqrt{2\pi}\ \xi}$$

$$\hat{f}(\xi) = \frac{2}{\xi}\frac{\sin \xi a}{\sqrt{2\pi}}, \quad a > 0. \quad \xi \in \mathbb{R}$$

$$f(x) = \mathcal{F}^{-1}\left(\frac{2}{\xi}\sin \xi a\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{2}{\xi}\frac{\sin \xi a}{\sqrt{2\pi}} \cdot e^{i\xi x}\, d\xi$$

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \xi a}{\xi} e^{i\xi x}\, d\xi$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \xi a}{\xi}\cos \xi x\, d\xi + \frac{i}{\pi}\int_{-\infty}^{\infty} \cancel{\frac{\sin \xi a}{\xi}\sin \xi x}\, d\xi$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin \xi a}{\xi}\cos \xi x\, d\xi$$

Allow $x \to 0$, to get $1 = f(0) = \frac{2}{\pi}\int_0^{\infty} \frac{\sin \xi a}{\xi}\, d\xi$

$$\Rightarrow \int_0^\infty \frac{\sin \xi a}{\xi}\, d\xi = \frac{\pi}{2}, \qquad \underline{a>0}.$$

<u>Example:</u> Find the Fourier transform of $f(x) = e^{-a^2x^2}$, $\underline{a>0}$.

<u>Sol:</u>

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{-i\xi x}\, dx, \qquad \underline{\xi \in \mathbb{R}}.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax+\frac{i\xi}{2a}\right)^2} e^{-\frac{\xi^2}{4a^2}}\, dx$$

$$= \frac{e^{-\frac{\xi^2}{4a^2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax+\frac{i\xi}{2a}\right)^2} dx$$

Let $ax + \frac{i\xi}{2a} = t$

$a\,dx = dt$

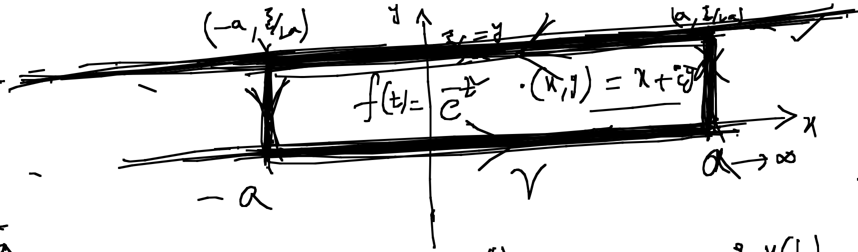
At $x = -\infty$, $t = -\infty + \frac{i\xi}{2a}$

At $x = \infty$, $t = \infty + \frac{i\xi}{2a}$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{\xi^2}{4a^2}}}{a} \int_{-\infty + \frac{i\xi}{2a}}^{\infty + \frac{i\xi}{2a}} e^{-t^2}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{\xi^2}{4a^2}}}{a} \cdot \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$\hat{f}(\xi) = \frac{1}{\sqrt{2}\, a} e^{-\frac{\xi^2}{4a^2}}, \quad \xi \in R.$$

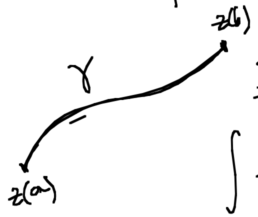


$x = t$, $z(t) = t + i\frac{\xi}{2a}$

$y = \xi/2a$

$-\infty < t < \infty$

$a \le t \le b$



$z(t) = x(t) + i\,y(t)$, $\frac{dz}{dt} = z'(t) \Rightarrow dz = z'(t)\,dt$

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt$$

$e^{-i\frac{\xi t}{a}}$

$$\int_{-\infty + \frac{i\xi}{2a}}^{\infty + \frac{i\xi}{2a}} e^{-z^2}\, dz = \int_{-\infty}^{\infty} e^{-\left(t + i\frac{\xi}{2a}\right)^2} dt$$

$z(t) = a + it$ $\quad e^{-(a+it)^2}$ $$= \int_{-\infty}^{\infty} e^{-t^2 + \frac{\xi^2}{4a^2}} \cos\left(\frac{\xi t}{a}\right) dt$$

$0 \le t \le \frac{\xi}{2a}$ $\quad \cancel{e^{-a^2} e^{-2iat} e^{t^2}}$ $$+ i \int_{-\infty}^{\infty} e^{-t^2 + \frac{\xi^2}{4a^2}} \sin\left(\frac{\xi t}{a}\right) dt$$

If $a=\frac{1}{\sqrt{2}}$, $f(x)=e^{-\frac{x^2}{2}}$

$\hat{f}(\xi)=e^{-\frac{\xi^2}{2}}$

$f(x+iy)=u(x,y)+iv(x,y)$

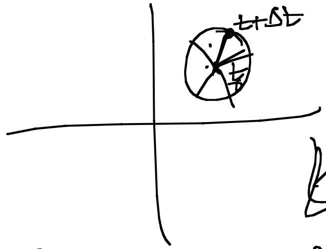
$\frac{\partial u}{\partial x}$ $\frac{\partial u}{\partial y}$ $\frac{\partial v}{\partial x}$ $\frac{\partial v}{\partial y}$ are cts

and C-R equations

Cauchy-Riemann

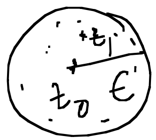
$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$



domain = open connected set

$$\left.\frac{df}{dz}\right|_{z=z_0} := \lim_{\Delta z} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \text{ exists}$$



$$e^{-z^2}=e^{-(x^2-y^2)-2xyi}$$

$$=e^{-(x^2-y^2)}\cos 2xy + i\, e^{-(x^2-y^2)}\sin 2xy$$

Cauchy Theorem: If $f(z)$ is analytic in a domain D, then

$$\oint_\gamma f(z)\,dz=0, \quad \forall \gamma \subset D.$$

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx \quad , \quad I = \int_{-\infty}^{\infty} e^{-y^2}\, dy$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \cdot \int_{-\infty}^{\infty} e^{-y^2}\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, \underline{dx\, dy}$$

$x = r\cos\theta, \quad y = r\sin\theta \qquad \underline{-\infty < x,\ y < \infty}$

$x^2 + y^2 = r^2$

$\underline{0 < r < \infty, \quad 0 \le \theta \le 2\pi}$

$dx\,dy = |\det(J)|\, dr\, d\theta$

$= r\, dr\, d\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta$

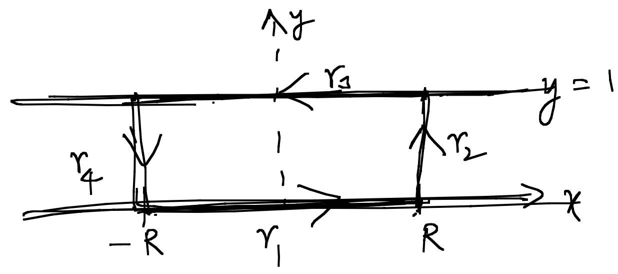
$= r$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

$$= \cancel{2}\pi \cdot \frac{-1}{\cancel{2}} e^{-r^2}\Big|_0^{\infty} = \pi.$$

$\Rightarrow I = \sqrt{\pi}$ ✓

* Evaluate $\displaystyle\int_{-\infty+i}^{\infty+i} e^{-z^2}\,dz$.



$D = \mathbb{C}$.

Let γ be the boundary of the rectangle $-R \le x \le R$, $0 \le y \le 1$.

$R > 0$.

✓ $e^{-z^2} = u(x,y) + i\,v(x,y)$

By Cauchy's theorem $\displaystyle\int_{\gamma} e^{-z^2}\,dz = 0$ ✓

$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$

$$\int_{\gamma_1} e^{-z^2}\,dz = \int_{-R}^{R} e^{-t^2}\,dt$$

γ_1: $y=0$, $-R \le x \le R$

$x = t$, $y = 0$ $\quad -R \le t \le R$

$z(t) = t$ $\quad z'(t) = 1$

$$\int_{R+i}^{-R+i} e^{-t^2}\,dt = \int_{\gamma_3} e^{-t^2}\,dt = \int_{R}^{-R} e^{-(t+i)^2}\,dt$$

γ_3 : $y=1$, $-R\le x\le R$

$z(t) = t+i$, $R\le t\le -R$

$$\int_{\gamma_2} e^{-t^2}\,dt = \int_0^1 e^{-(R+it)^2}\, i\,dt$$

γ_2 : $x=R$, $0\le y\le 1$

$z(t) = R+it$, $0\le t\le 1$

$z'(t)\,dt = i\,dt$

$$\int_a^b x(t)+iy(t)\,dt = \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt$$

$$= ie^{-R^2}\int_0^1 e^{t^2-2iRt}\,dt \to 0 \text{ as } R\to\infty$$

$$\int_{\gamma_4} e^{-t^2}\,dt = \int_1^0 e^{-(R+it)^2}\, i\,dt$$

γ_4 : $x=-R$, $1\le y\le 0$

$z(t) = -R+it$, $1\le t\le 0$

$z'(t)\,dt = i\,dt$

$$= -ie^{-R^2}\int_0^1 e^{t^2-2iRt}\,dt \longrightarrow 0 \text{ as } R\to\infty$$

As $R\to\infty$, $\int_\gamma e^{-t^2}\,dt \longrightarrow \int_{\infty+i}^{-\infty+i} e^{-t^2}\,dt + \int_{-\infty}^{\infty} e^{-t^2}\,dt = 0$ ✓

$$\Rightarrow \int_{-\infty+i}^{\infty+i} e^{-z^2}\, dz = \int_{-\infty}^{\infty} e^{-t^2}\, dt \quad \checkmark$$

Properties of Fourier transform:

1. Linear property

$$\hat{f}(\xi) = \widehat{f_1 + c f_2}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(f_1(x) + c f_2(x)\right) e^{-i\xi x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) e^{-i\xi x}\, dx + c \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(x) e^{-i\xi x}\, dx$$

$$\hat{f}(\xi) = \hat{f}_1(\xi) + c\, \hat{f}_2(\xi).$$

2. If $g(x) = f(ax)$, then $\hat{g}(\xi) = \frac{1}{|a|} \cdot \hat{f}\left(\frac{\xi}{a}\right)$.

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{-i\xi x}\, dx$$

$$|a| = \begin{cases} a, & a > 0 \\ -a, & a < 0 . \end{cases}$$

$ax = t$

$dx = \frac{dt}{a}$ ✓

$$= \frac{1}{|a|} \underline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\frac{\xi}{a}t}\, dt}$$

$$= \frac{1}{|a|} \hat{f}\left(\frac{\xi}{a}\right), \quad \text{if } |a| \neq 0 .$$ ✓

3. If $g(x) = f(x-a)$, then $\hat{g}(\xi) = \underline{e^{-i\xi a}\, \hat{f}(\xi)}$ ✓

$$\hat{g}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\xi(a+t)}\, dt$$

$x - a = t$

$dx = dt$

$$= e^{-i\xi a}\, \hat{f}(\xi)$$

4. If $g(x) = f(x)\, \underline{e^{-iax}}$, then $\hat{g}(\xi) = \hat{f}(\xi + a)$.

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iax}\, e^{-i\xi x}\, dx = \hat{f}(\xi + a)$$

$\hat{f} : \mathbb{R} \to \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$ ✓

5. If $g(x) = f(x) \cos ax$, then $\hat{g}(\xi) = \dfrac{\hat{f}(\xi - a) + \hat{f}(\xi + a)}{2}$ ✓

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax\, e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \frac{e^{iax} + e^{-iax}}{2} \cdot e^{-i\xi x}\, dx$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i(\xi - a)x}\, dx + \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i(\xi + a)x}\, dx$$

$$= \frac{1}{2} \hat{f}(\xi - a) + \frac{1}{2} \hat{f}(\xi + a).$$

6. If $g(x) = f(x) \sin ax$, then $\hat{g}(\xi) = -\dfrac{i}{2} \left[\hat{f}(\xi - a) - \hat{f}(\xi + a) \right]$.

Cor: If $g(x) = f(x) \cos ax$, then $F_c(g(x))(\xi) = \frac{1}{2} \left[F_c(f(x))(\xi - a) + F_c(f(x))(\xi + a) \right]$ ✓

If $g(x) = f(x)\sin ax$, then $F_c(g(x))(\xi) = \frac{1}{2}\left[F_s(f(x))(a-\xi) + F_s(f(x))(\xi+a)\right]$

If $g(x) = f(x)\cos ax$, then $F_s(g(x))(\xi) = \frac{1}{2}\left[F_s(f(x))(\xi-a) + F_s(f(x))(\xi+a)\right]$

If $g(x) = f(x)\sin ax$, then $F_s(g(x))(\xi) = \frac{1}{2}\left[F_c(f(x))(\xi-a) - F_c(f(x))(\xi+a)\right]$

7. If $f'(x)$ is piecewise continuously differentiable function and f and $f'(x)$ are absolutely integrable in $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} |f'(x)|\,dx < \infty \overset{?}{\Rightarrow} \int_{-\infty}^{\infty} |f(x)|\,dx < \infty$$

then $\widehat{f'}(\xi) = i\xi \hat{f}(\xi)$.

$$\widehat{f'}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-i\xi x}\,dx = \cancel{\frac{1}{\sqrt{2\pi}} f(x) e^{-i\xi x}\Big|_{-\infty}^{\infty}} + \frac{i\xi}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\,dx$$

$$= i\xi \cdot \hat{f}(\xi).$$

$$\hat{f}(\xi) = \frac{1}{i\xi} \widehat{f'}(\xi)$$

$$\Rightarrow \widehat{f''}(\xi) = i\xi\,\widehat{f'}(\xi) = (i\xi)^2\,\hat{f}(\xi)$$

$$\Rightarrow \hat{f}(\xi) = \frac{1}{(i\xi)^2}\,\widehat{f''}(\xi).$$

$$\vdots$$

$$\hat{f}(\xi) = \frac{1}{(i\xi)^n}\cdot\widehat{f^{(n)}}(\xi).$$

8. If $f(x)$ is piecewise smooth function in $(0,\infty)$ and $\int_0^\infty |f|\,dx < \infty$, $\int_0^\infty |f'(x)|\,dx < \infty \cdots \int_0^\infty |f^{(n)}(x)|\,dx < \infty$ then

$$F_c(f'(x))(\xi) = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos\xi x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\,f(x)\cos\xi x\Big|_0^\infty + \xi\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\xi x\,dx$$

$$F_c(f'(x))(\xi) = -\sqrt{\frac{2}{\pi}}\, f(0) + \xi\, F_c(f(x))(\xi).$$

Ex: Find $F_s(f'(x))(\xi)$.

$F_c(f''(x))(\xi)$

$F_s(f''(x))(\xi)$.

Defn: (Convolution product) Let $f(x)$ and $g(x)$ be two absolutely integrable functions in $(-\infty, \infty)$.

Then
$$f * g(x) := \int_{-\infty}^{\infty} f(x-y) \cdot g(y)\, dy.$$

9. If $f(x)$ and $g(x)$ are two absolutely integrable functions in $(-\infty, \infty)$,

then
$$\widehat{f*g(x)}(\xi) = \sqrt{2\pi}\, \hat{f}(\xi) \cdot \hat{g}(\xi).$$

$$\widehat{f*g(x)}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f*g(x)\, e^{-i\xi x}\, dx, \quad \xi \in \mathbb{R}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy\, e^{-i\xi x}\, dx. \checkmark$$

$$\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{f(x-y)\, g(y)\, e^{-i\xi x}}_{h(x,y)}\, dy\, dx\right| \le \underline{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x-y)\cdot g(y)|\, dy\, dx} \underline{< \infty}$$

$$\left|\underline{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\, dx\, g(y)\, dy}\right| \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x)||g(y)|\, dx\, dy \;\underline{\underline{< \infty}}$$

<u>Fubini's theorem</u>: If $f(x)$ and $g(y)$ are absolutely integrable functions in $(-\infty, \infty)$.

and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x)\cdot g(y)|\, dx\, dy < \infty$, Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x) g(y)|\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x)|\, dx \cdot |g(y)|\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(y)|\, dy\, |f(x)|\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y)\, g(y)\, e^{-i\xi x}\, dx\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y)\, e^{-i\xi x}\, dx \cdot g(y)\, dy$$

Let $x-y=t$, $dx = dt$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{-i\xi(y+t)}\, dt\, g(y)\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underline{\left(\int_{-\infty}^{\infty} f(t)\, e^{-i\xi t}\, dt \right)}\, g(y)\, e^{-i\xi y}\, dy$$

$$\widehat{f * g(x)}(\xi) = \sqrt{2\pi}\, \hat{f}(\xi) \cdot \underline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)\, e^{-i\xi y}\, dy} = \sqrt{2\pi}\, \hat{f}(\xi) \cdot \hat{g}(\xi).$$

10. <u>parsevals identity</u> $\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi$ ✓ $\qquad |f(x)|^2 = f(x) \cdot \overline{f(x)}$.

if $f(x)$ is absolutely integrable function in $(-\infty, \infty)$.

$$\widehat{f * g(x)}\,(\xi) = \sqrt{2\pi}\; \hat{f}(\xi) \cdot \hat{g}(\xi).$$

$$\mathcal{F}^{-1}\left(\widehat{f * g(x)}\,(\xi)\right)(x) = \cancel{\sqrt{2\pi}} \cdot \frac{1}{\cancel{\sqrt{2\pi}}} \int_{-\infty}^{\infty} \hat{f}(\xi)\, \hat{g}(\xi) \cdot e^{i\xi x} \, d\xi .$$

$$\int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy = f * g(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot \hat{g}(\xi)\, e^{i\xi x} \, d\xi .$$

put $x = 0$, $\displaystyle\int_{-\infty}^{\infty} f(-y)\, g(y)\, dy = \int_{-\infty}^{\infty} \hat{f}(\xi)\, \hat{g}(\xi)\, d\xi .$

$$\int_{-\infty}^{\infty} f(t)\, g(-t)\, dt = \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot \underline{\hat{g}(\xi)}\, d\xi$$ ✓

Let $\underline{g(-t)} = \overline{f(t)}$, then $\hat{g}(\xi) = \underline{\widehat{g(t)}}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t)\, e^{-i\xi t}\, dt = \overline{\overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-t)}\, e^{-i\xi t}\, dt}}$

$g(t) = \overline{f(-t)}$ ✓

$$= \overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-t)\, e^{-i(\xi)t}\, dt}$$

$-t = x$

$$= \overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx}$$

$$\hat{g}(\xi) = \underline{\overline{\hat{f}(\xi)}}.$$

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot \overline{\hat{f}(\xi)}\, d\xi$$

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi}$$ ✓

11.. Rieman- Lebesgue Lemma: If $f(x)$ is absolutely integrable function in $(-\infty,\infty)$ and $f(x)$ is piecewise continuous function, then

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0.$$

Proof:

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

$$= \lim_{A\to\infty} \int_{-A}^{A} |f(x)|\, dx < \infty, \quad A \in (-\infty,\infty)$$

$$\int_{-\infty}^{\infty} |f(x)|\, dx = \int_{-\infty}^{-A} |f(x)|\, dx + \int_{A}^{\infty} |f(x)|\, dx + \int_{-A}^{A} |f(x)|\, dx.$$

:

$$\left| \int_{-A}^{A} |f(x)|\, dx - \int_{-\infty}^{\infty} |f(x)|\, dx \right| = \underline{\int_{-\infty}^{-A} |f(x)|\, dx + \int_{A}^{\infty} |f(x)|\, dx} < \epsilon, \quad \text{for some } A \in (-\infty, \infty)$$

$B > A$, $\frac{n}{B} > N$

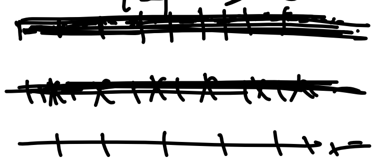
$w_0 = \frac{2\pi}{2A} = \frac{\pi}{A}$

$$f(x),\quad x \in [-A, A],\quad \underline{\lim_{\pi\frac{n}{A} \to \infty}} c_n = \underline{\lim_{n\to\infty} \frac{1}{2A} \int_{-A}^{A} f(x)\, e^{-in\frac{\pi}{A}x}\, dx} = 0$$

(Riemann-Lebesgue Lemma for periodic signals)

$$\underline{\left| \int_{-A}^{A} f(x)\, e^{-i \overbrace{n\frac{\pi}{A}}^{\xi} x}\, dx \right| < \epsilon}, \quad \text{for } \underline{\overbrace{\pi\frac{n}{A}}^{\xi} > N^{>0}} \text{ for some } \underline{N \in \mathbb{N}}.$$

$|\xi| \to \infty$.



$$\Rightarrow \left| \int_{-A}^{A} f(x)\, e^{-i\xi x}\, dx \right| < \epsilon, \quad \text{for } |\xi| > N > 0.$$

$$\left|\hat{f}(\xi)\right| = \frac{1}{\sqrt{2\pi}}\left|\int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx\right| \leq \frac{1}{\sqrt{2\pi}}\left|\int_{-\infty}^{-A} + \int_{A}^{\infty} f(x)\, e^{-i\xi x}\, dx\right| + \frac{1}{\sqrt{2\pi}}\left|\int_{-A}^{A} f(x)\, e^{-i\xi x}\, dx\right|$$

$$\leq \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{-A} |f(x)|\, dx + \int_{A}^{\infty} |f(x)|\, dx\right) + \frac{1}{\sqrt{2\pi}}\epsilon\,.$$

$$\left|\hat{f}(\xi)\right| \leq \frac{1}{\sqrt{2\pi}}\cdot 2\epsilon = \sqrt{\frac{2}{\pi}}\,\epsilon\,, \quad |\xi| > N\,.$$

$$\Rightarrow \lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$$

11. (Riemann – Lebesgue Lemma)

If $f(x)$ is absolutely integrable and piecewise continuous in $(-\infty, \infty)$, then

$\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$, i.e., given $\epsilon > 0$, $|\hat{f}(\xi)| < \epsilon$, $|\xi| > K > 0$, for some $K \in \mathbb{R}$ ✓

proof:

$$|\hat{f}(\xi)| = \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx \right|$$

$$= \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-A} f(x) e^{-i\xi x} dx + \frac{1}{\sqrt{2\pi}} \int_{A}^{\infty} f(x)\, e^{-i\xi x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f(x)\, e^{-i\xi x}\, dx \right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{-A} |f(x)|\, dx + \int_{A}^{\infty} |f(x)|\, dx \right) + \frac{1}{\sqrt{2\pi}} \left| \int_{-A}^{A} f(x)\, e^{-i\xi x}\, dx \right| \leq \left(\frac{1}{\sqrt{2\pi}} + \frac{2(A+1)}{\sqrt{2\pi}} \right) \epsilon < L\epsilon, \quad \forall\, |\xi| > K = \max\{k_1, k_2\}$$

where L is a finite real #.

$$\lim_{\substack{A+1\to\infty \\ A\to\infty}} \int_{-A+1}^{A+1} |f(x)| = \int_{-\infty}^{\infty} |f(x)|\,dx,$$

given $\epsilon > 0$,

$$\Rightarrow \left| \int_{-\infty}^{-A+1} |f(x)|\,dx + \int_{A+1}^{\infty} |f(x)|\,dx \right| < \epsilon, \quad \text{if } A+1 > K_1 \text{ for some } K_1 \in \mathbb{R}.$$

$\supset [-B, B]$

$f(x),\ x \in [-A+1, A+1]$

Riemann-Lebesgue Lemma for periodic signals

$$\lim_{|n|\to\infty} C_n = \lim_{|n|\to\infty} \frac{1}{2(A+1)} \int_{-A+1}^{A+1} f(x)\, e^{-i \frac{n\pi}{A+1} x}\, dx = 0$$

ξ

$\left\{ \left| \frac{n\pi}{A+1} \right| \right\} \to \infty$ as $n \to \infty$

If $B = A + y$, $y \in [0, 1]$

then $$\left| \int_{-\infty}^{-B} |f(x)|\,dx + \int_{B}^{\infty} |f(x)|\,dx \right| < \epsilon, \quad \text{if } B > K_1 \text{ for some } K_1 \in \mathbb{R}$$

$\Rightarrow \left\{ \frac{n\pi}{B} \right\} \to \infty$ as $n \to \infty$

$$\left| \frac{1}{2(A+1)} \int_{-(A+1)}^{A+1} f(x)\, e^{-i \xi x}\, dx \right| < \epsilon, \quad \text{if } |\xi| > K_2 > 0, \text{ for some } K_2 \in \mathbb{N}.$$

Theorem: (Dominated convergence theorem)

Let $f_r(x)$, $r \in \mathbb{R}$ be a family of piecewise absolutely integrable continuous functions.

If $|f_r(x)| \leq g(x)$, $\forall\, x \in \mathbb{R}$ with $\int_{-\infty}^{\infty} g(x)\, dx < \infty$.

and $\lim_{r \to 0} f_r(x) = f(x)$, $\forall\, x \in \mathbb{R}$, then

$$\lim_{r \to 0} \int_{-\infty}^{\infty} f_r(x)\, dx = \int_{-\infty}^{\infty} \lim_{r \to 0} f_r(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx. \checkmark$$

Riemann–Lebesgue Lemma: If $f(x)$ is piecewise continuous and absolutely integrable function, then $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$ ✓

Proof:

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, dx \quad , \quad \xi \in \mathbb{R}$$

Let $u = x - \frac{\pi}{\xi}$, then

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u+\pi/\xi)\, e^{-i\xi(u+\pi/\xi)}\, du$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u+\pi/\xi)\, e^{-i\xi u}\, du$$

$$\lim_{|\xi|\to\infty} \left(|\hat{f}(\xi)| = \lim_{|\xi|\to\infty} \frac{1}{2} \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} \left[\underline{f(u) - f(u+\pi/\xi)} \right] e^{-i\xi u}\, du \right| \right.$$

$$\left| \underline{f(u) - f(u+\pi/\xi)} \right| \leq |f(u)| + |f(u+\pi/\xi)|$$

$$\lim_{|\xi|\to\infty} f(u) - f\left(u+\tfrac{\pi}{\xi}\right) = 0, \ \forall\, u \in \mathbb{R}. \checkmark$$

$$\Rightarrow \quad \underline{\lim_{|\xi|\to\infty} |\hat{f}(\xi)|} = \frac{1}{2\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} \lim_{|\xi|\to\infty} \left(f(u) - f\left(u+\tfrac{\pi}{\xi}\right)\right) e^{-i\xi u} \, du \right|$$

$$\leq \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \underline{\left| \lim_{|\xi|\to\infty} \left(f(u) - f\left(u+\tfrac{\pi}{\xi}\right)\right) \right|} \, du$$

$$\leq 0$$

$$\boxed{\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0} \ \checkmark$$

:

12. If $f(x)$ is absolutely integrable function and piece-wise continuous function in $(-\infty, \infty)$, then

$\hat{f}(\xi)$ is a continuous function. ✓

Proof:

$$\lim_{h \to 0} \hat{f}(\xi+h) - \hat{f}(\xi) = \lim_{h \to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[e^{-i(\xi+h)x} - e^{-i\xi x} \right] dx$$

$$= \lim_{h \to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x} \left(e^{-ihx} - 1 \right) dx .$$

$\lim_{h \to 0} f(x) \left[e^{-ihx} - 1 \right] = 0, \ \forall\, x \in \mathbb{R}$. and

$$\left| f(x) \left(e^{-ihx} - 1 \right) \right| \leq |f(x)| + |f(x)| = 2\,|f(x)|$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{h\to 0} f(x)\, e^{-i\xi x}\left(e^{-ihx}-1\right) dx$$

$$= 0 \checkmark$$

$\Rightarrow$ $\hat{f}(\xi)$ is a continuous function in $(-\infty, \infty)$.

Fourier integral theorem:

If $f(x)$ is an absolutely integrable function and piecewise smooth function in $(-\infty, \infty)$, then

$$\lim_{r\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-r}^{r} \hat{f}(\xi)\, e^{i\xi x}\, d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\, e^{-i\xi y}\, dy\, e^{i\xi x}\, d\xi$$

$$= \frac{1}{2}\left(f(x^+) + f(x^-)\right), \quad \forall\, x \in (-\infty, \infty)$$

proof: Observe that $\displaystyle\int_{-r}^{r} e^{i\xi(x-y)}\,d\xi = \frac{e^{i\xi(x-y)}}{i(x-y)}\Bigg|_{\xi=-r}^{\xi=r} = \frac{1}{i(x-y)}\left(e^{ir(x-y)} - e^{-ir(x-y)}\right)$

$$= \frac{2\sin\left(r(x-y)\right)}{(x-y)} \checkmark$$

$$\underline{\frac{1}{\sqrt{2\pi}}\int_{-r}^{r} \hat{f}(\xi)\, e^{i\xi x}\,d\xi} = \frac{1}{2\pi}\int_{-r}^{r}\int_{-\infty}^{\infty} f(y)\, e^{i\xi(x-y)}\,dy\,d\xi.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y) \underline{\int_{-r}^{r} e^{i\xi(x-y)}\,d\xi}\,dy.$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\,\frac{\sin r(x-y)}{(x-y)}\,dy.$$

$$x - y = t \qquad y = x - t$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin nt}{t} \cdot f(x-t)\, dt$$

$$= \frac{1}{\pi} \left(f * g_n (x) \right).$$

where $g_n(x) = \frac{\sin nx}{x} = \frac{\sin nx}{nx} \cdot n = \frac{1}{1/n} \cdot \frac{\sin x/_{1/n}}{x/_{1/n}}$.

$$= \frac{1}{\pi} \left(f * g_{1/n} \right)(x),$$

where $g_{1/n}(x) = \frac{1}{1/n}\, g\left(\frac{x}{1/n}\right)$, with $g(x) = \frac{\sin x}{x}$.

:

we note that $\displaystyle\int_0^\infty \frac{\sin rx}{x}\,dx = \int_{-\infty}^{0} \frac{\sin rx}{x}\,dx = \frac{\pi}{2}$, if $r > 0$

$$\lim_{r\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-r}^{r} e^{i\xi x} \hat{f}(\xi)\, d\xi - \frac{1}{2}\left(f(x^+) + f(x^-)\right) =$$

$$= \lim_{r\to\infty} \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin ry}{y} f(x-y)\, dy - \frac{\cancel{2}}{\pi}\cdot\frac{1}{\cancel{2}} f(x^+) \int_0^\infty \frac{\sin ry}{y}\, dy - \frac{\cancel{2}}{\pi}\cdot\frac{1}{\cancel{2}} f(x^-) \cdot \int_{-\infty}^{0} \frac{\sin ry}{y}\, dy \right\}$$

$$= \lim_{r\to\infty} \frac{1}{\pi} \left[\int_0^\infty \frac{\sin ry}{y} \left(f(x-y) - f(x^+)\right) dy + \int_{-\infty}^{0} \frac{\sin ry}{y} \left(f(x-y) - f(x^-)\right) dy \right.$$

$$= 0$$

Consider $\displaystyle\int_0^\infty \frac{\sin ry}{y}\left(f(x-y)-f(x^+)\right)dy = \left(\int_0^K + \int_K^\infty \frac{\sin ry}{y}\left(f(x-y)-f(x^+)\right)dy\right)$, for any $K>0$

$ry = t$, $dy = \frac{dt}{r}$

If $K \geq 1$ $\displaystyle\left|\int_K^\infty \frac{\sin ry}{y} f(x-y)\,dy\right| \leq \int_K^\infty |f(x-y)|\,dy$ and $\displaystyle\int_K^\infty \frac{\sin ry}{y} f(x^+)\,dy = f(x^+)\int_{Kr}^\infty \frac{\sin t}{t}\,dt.$ ✓

If $K \to \infty$, $\displaystyle\int_K^\infty \frac{\sin ry}{y}\left(f(x-y) - f(x^+)\right)dy \longrightarrow 0.$

$$\int_0^K \frac{\sin ry}{y}\left(f(x-y) - f(x^+)\right)dy = \int_0^K \sin ry \cdot g(y)\,dy$$

where $\dfrac{f(x-y) - f(x^+)}{y} =: g(y)$, $y \in [0, K]$ ✓

$$= \lim_{y \to 0} \frac{f(x-y) - f(x^+)}{y} = f'(x^+) < \infty$$

$$= \int_0^k g(y) \, \frac{e^{iry} - e^{-iry}}{2i} \, dy$$

$$= \frac{i}{2}\left(C_r - C_{-r}\right)$$

For large 'r', $\left|\frac{i}{2}\left(C_r - C_{-r}\right)\right| < \epsilon.$

$$\left| \int_0^k \frac{\sin ry}{y} \left(f(x-y) - f(x^+)\right) dy \right| < \epsilon, \quad r > R > 0.$$

$$\Rightarrow \left| \int_0^\infty \frac{\sin ry}{y} \left(f(x-y) - f(x^+) \right) dy \right| < 2\epsilon, \quad \text{for } r > R.$$

$$\lim_{r\to\infty} \int_0^\infty \frac{\sin ry}{y} \left(f(x-y) - f(x^+) \right) dy = 0 \ \checkmark$$

:

Application of Fourier transform:

To solve ordinary differential equations that are linear.

$$y(x) = \underline{c_1 y_1 + c_2 y_2 + \cdots + c_n y_n} + y_p(x)$$

$$Ly(x) \equiv a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x),$$

$-\infty < x < \infty$.

$a_n, a_{n-1}, \ldots, a_1, a_0$ are constants.

Apply Fourier transform,

$$\widehat{a_n \frac{d^n y}{dx^n}}(\xi) + \widehat{a_{n-1} \frac{d^{n-1} y}{dx^{n-1}}}(\xi) + \cdots + \widehat{a_1 \frac{dy}{dx}}(\xi) + \widehat{a_0 y}(\xi) = \hat{f}(\xi),$$

$\xi \in \mathbb{R}$.

$$a_n (i\xi)^n \hat{y}(\xi) + a_{n-1} (i\xi)^{n-1} \hat{y}(\xi) + \cdots + a_1 (i\xi) \hat{y}(\xi) + a_0 \hat{y}(\xi) = \hat{f}(\xi).$$

$$\hat{y}(\xi) \underbrace{\left(a_n (i\xi)^n + \cdots + a_1 \cdot i\xi + a_0\right)}_{= P_n(\xi)} = \hat{f}(\xi).$$

$$\Rightarrow \quad \hat{y}(\xi) = \frac{\hat{f}(\xi)}{P_n(\xi)}, \quad \xi \in \mathbb{R}$$

$$y(x) = c_1 y_1 + c_2 y_2 + y_p(x)$$

$$\Rightarrow \quad \boxed{y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{P_n(\xi)} \cdot e^{i\xi x} \, d\xi.} \longrightarrow \boxed{\text{if } f = 0,\ y(x) = 0}$$

Example: Solve $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{-x}, \quad x > 0$

$$y(0) = 0 = y'(0).$$

$y(x)$.

$\left.\begin{array}{l} y(0)=0 \\ y'(0)=0 \end{array}\right\}$ 0

$x \in (-\infty, \infty)$.

Solution: Extend $y(x), \ x \in (-\infty, \infty)$.

$\underline{x<0}$

$\left.\begin{array}{l} y'' + 3y' + 2y = 0 \\ y(0) = 0 = y'(0) \end{array}\right\} y(x) = 0 \ \forall x \in (-\infty, 0)$

Equation now is $\quad \dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \le 0 \end{cases} = e^{-x} H(x), \quad x \in (-\infty, \infty)$

Solution is $y(x) = 0, \ \forall \ x \leq 0$.

$$\widehat{y''}(\xi) + 3\widehat{y'}(\xi) + 2\hat{y}(\xi) = \widehat{e^{-x}H(x)}(\xi)$$

$$\left[(i\xi)^2 + 3(i\xi) + 2\right]\hat{y}(\xi) = \frac{1}{\sqrt{2\pi}}\frac{1}{1+i\xi}$$

$$\hat{y}(\xi) = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{1+i\xi}\cdot\frac{1}{(i\xi+1)(i\xi+2)}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{1}{1+i\xi} + \frac{1}{(1+i\xi)^2} + \frac{1}{2+i\xi}\right].$$

$$\Rightarrow y(x) = -e^{-x}H(x) + xe^{-x}H(x) + e^{-2x}H(x), \ x \in (-\infty, \infty).$$

$y(x) = 0, \ x \leq 0$

$y(0) = 0 = y'(0)$

$$\boxed{y(x) = -e^{-x} + xe^{-x} + e^{-2x}, \ x > 0}$$

$y(0) = -1 + 1 = 0$, $y'(x) = e^{-x} - xe^{-x} + e^{-x} - 2e^{-2x}$

$y'(0) = 1 + 1 - 2 = 0$

R.H.S $= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x}e^{-i\xi x}dx = \frac{1}{\sqrt{2\pi}}\left.\frac{-e^{-x(1+i\xi)}}{1+i\xi}\right|_0^\infty = \frac{1}{\sqrt{2\pi}}\frac{1}{1+i\xi}$

$x^2 + 3x + 2$

$(x+1)(x+2)$

$$\boxed{\widehat{x^k e^{-x}H(x)}(\xi) = \frac{1}{\sqrt{2\pi}}\frac{1}{(1+i\xi)^{k+1}}, \ \forall k \in \mathbb{N}}$$

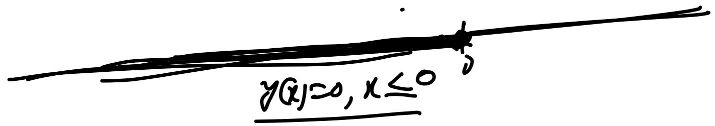
$$\widehat{xe^{-x}H(x)}(\xi) = \frac{1}{\sqrt{2\pi}}\int_0^\infty x\, e^{-x} e^{-i\xi x}dx$$

$$= \cancel{\frac{1}{\sqrt{2\pi}}\left.\frac{-e^{-x(1+i\xi)}}{(1+i\xi)}x\right|_0^\infty} + \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-x(1+i\xi)}}{(1+i\xi)}dx$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{(1+i\xi)^2}$$

Example: Solve $y'' - 4y' + 4y = x e^{-x}, \quad x > 0.$

$$y(0) = 0 = y'(0).$$



Soln: Extend: $y(x), \quad x \in (-\infty, \infty)$

$$y'' - 4y' + 4y = x e^{-x} H(x), \quad x \in (-\infty, \infty)$$

$$(i\xi)^2 \hat{y}(\xi) - 4(i\xi)\hat{y}(\xi) + 4\hat{y}(\xi) = \frac{1}{\sqrt{2\pi}} \frac{1}{(1+i\xi)^2}$$

$$\hat{y}(\xi) = \frac{1}{\sqrt{2\pi}\,(1+i\xi)^2} \frac{1}{(i\xi - 2)^2}.$$

$$\hat{y}(\xi) = \frac{1}{\sqrt{2\pi}} \left[\frac{2/27}{1+i\xi} + \frac{1/9}{(1+i\xi)^2} + \frac{-2/27}{(i\xi - 2)} + \frac{1/9}{(i\xi-2)^2} \right]$$

$$y(x) = \frac{2}{27} e^{-x} H(x) + \frac{1}{9} x e^{-x} H(x) - \frac{2}{27} e^{2x} H(x) + \frac{1}{9} x e^{2x} H(x), \quad x \in (-\infty, \infty).$$

$$\boxed{y(x) = \frac{2}{27} e^{-x} + \frac{1}{9} x e^{-x} - \frac{2}{27} e^{2x} + \frac{1}{9} x e^{2x}, \quad x > 0.}$$

Verification

$$\begin{cases} y(0) = 0, \quad y'(x)\Big|_{x=0} = -\frac{2}{27} e^{-x} + \frac{1}{9} e^{-x} - \frac{1}{9} x e^{-x} - \frac{2 \times 2}{27} e^{2x} + \frac{1}{9} e^{2x} + \frac{2}{9} x e^{2x}\Big|_{x=0} \\ y'(0) = -\frac{2}{27} + \frac{1}{9} - \frac{4}{27} + \frac{1}{9} = -\frac{\cancel{6}^{2}}{\cancel{27}_{9}} + \frac{2}{9} = 0 \end{cases}$$

Example: Solve $y'' - 4y' + 5y = 1, \quad x > 0.$

$$y(0) = 0 = y'(0)$$

Soln: Extend to $x \in (-\infty, \infty)$ to get

$$y'' - 4y' + 5y = H(x), \quad x \in (-\infty, \infty).$$

$$\hat{H}(\xi) = \sqrt{\frac{\pi}{2}}\,\delta(\xi) + \underline{\frac{1}{\sqrt{2\pi}}\frac{1}{i\xi}}.$$

$$\left[(i\xi)^2 - 4(i\xi) + 5\right]\hat{y}(\xi) = \sqrt{\frac{\pi}{2}}\,\delta(\xi) + \frac{1}{\sqrt{2\pi}}\frac{1}{i\xi}.$$

$$x^2 - 4x + 5 = 0$$

$$x = \frac{4 \pm \sqrt{16-20}}{2} = \underline{2 \pm i}$$

$$\hat{y}(\xi) = \sqrt{\frac{\pi}{2}}\frac{\delta(\xi)}{(i\xi-2-i)(i\xi-2+i)} + \frac{1}{\sqrt{2\pi}}\frac{1}{i\xi}\cdot\frac{1}{(i\xi-2-i)(i\xi-2+i)}.$$

$$= \sqrt{\frac{\pi}{2}}\frac{\delta(\xi)}{(i\xi-2-i)(i\xi-2+i)} + \frac{1}{\sqrt{2\pi}}\left[\frac{1/5}{i\xi} + \frac{1/(-2+4i)}{i\xi-2-i} + \frac{1/(-2-4i)}{i\xi-2+i}\right]$$

$$y(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{\pi}{2}}\frac{\delta(\xi)\,e^{i\xi x}}{(i\xi-2-i)(i\xi-2+i)}\,d\xi + \frac{1}{5}\left(H(x)-\frac{1}{2}\right) + \frac{1}{(-2+4i)}\cdot e^{(2+i)x}H(x) - \frac{1}{2+4i}e^{(2-i)x}H(x), \quad x \in \mathbb{R}.$$

$$y(x) = \frac{1}{5}H(x) + \frac{e^{(2+i)x}}{-2+4i}H(x) - \frac{1}{2+4i}e^{(2-i)x}H(x), \quad x \in (-\infty, \infty).$$

$$\Rightarrow \quad y(x) = \frac{1}{5} + e^{2x}\left[\frac{e^{ix}}{-2+4i} - \frac{e^{-ix}}{2+4i}\right], \quad x > 0$$

$$= \frac{1}{5} + e^{2x}\left[\frac{e^{ix}(-2-4i)}{20} - \frac{e^{-ix}(2-4i)}{20}\right]$$

$$= \frac{1}{5} - \frac{e^{2x}}{10}\left[e^{ix}(1+2i) + e^{-ix}(1-2i)\right]$$

$$= \frac{1}{5} - \frac{e^{2x}}{5}\cos x - \frac{e^{2x}}{\cancel{10}\,5} \cdot \cancel{2}i \cdot 2i \sin x$$

$$\boxed{y(x) = \frac{1}{5} - \frac{e^{2x}}{5}\cos x + \frac{2}{5}e^{2x}\sin x; \quad x > 0.}$$

$$y(0) = 0, \quad y'(x) = \sin x\,\frac{e^{2x}}{5} - \cancel{\frac{2}{5}e^{2x}\cos x} + \cancel{\frac{2}{5}\cos x\, e^{2x}} + \frac{4}{5}e^{2x}\sin x$$

$$\underline{y(0) = 0}. \quad y'(0) = 0$$

Example: Solve $y'' + 3y' + 2y = e^{-x}, \quad x > 0.$

$$y(0) = 1, \quad y'(0) = 2.$$

$y(x)$

$0 \qquad \infty$

$\lim_{x\to\infty} y(x) = 0$

Soln: Let us solve this by Fourier cosine transform.

$$F_c\left(\frac{d^2y}{dx^2}\right)(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d^2y}{dx^2} \cos \xi x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[y'(x) \cos \xi x \Big|_0^\infty + \xi \int_0^\infty y'(x) \sin \xi x \, dx \right]$$

$$= -\sqrt{\frac{2}{\pi}}\, 2 + \sqrt{\frac{2}{\pi}}\, \xi \left[\cancel{y(x) \sin \xi x \Big|_0^\infty} - \xi \int_0^\infty y(x) \cos \xi x \, dx \right]$$

$$= -2\sqrt{\frac{2}{\pi}} - \xi^2 F_c(y(x))(\xi).$$

$$-2\sqrt{\frac{2}{\pi}} - \xi^2 F_c(y(x))(\xi) - 3\sqrt{\frac{2}{\pi}} + 3\xi \underline{F_s(y(x))(\xi)} + 2F_c(y(x))(\xi) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\xi^2}, \quad \xi > 0$$

$$3\xi F_s(y(x))(\xi) + \underline{F_c(y(x))(\xi)}(2-\xi^2) = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+\xi^2} + 5\right) \text{———} 1$$

Also, apply Fourier sine transform to the equation to get,

$$F_s(y''(x))(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d^2y}{dx^2} \sin \xi x \, dx = \sqrt{\frac{2}{\pi}}\left[y'(x) \sin \xi x \Big|_0^\infty - \xi \int_0^\infty y'(x) \cos \xi x\right]$$

$$= -\xi \sqrt{\frac{2}{\pi}}\left[y(x) \cos \xi x \Big|_0^\infty + \xi \int_0^\infty y(x) \sin \xi x \, dx\right]$$

$$= \xi\sqrt{\frac{2}{\pi}} - \xi^2 F_s(y(x))(\xi).$$

:

$$\xi\sqrt{\frac{2}{\pi}} - \xi^2 F_s(y(x))(\xi) - 3\xi F_c(y(x))(\xi) = \sqrt{\frac{2}{\pi}}\frac{\xi}{1+\xi^2}, \quad \xi>0$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin\xi x\,dx = \sqrt{\frac{2}{\pi}}\frac{\xi}{a^2+\xi^2}$$

$$\xi^2 F_s(y(x))(\xi) + 3\xi F_c(y(x))(\xi) = \sqrt{\frac{2}{\pi}}\left(\frac{\xi}{1+\xi^2} - \xi\right)$$

$$= -\sqrt{\frac{2}{\pi}}\frac{\xi^3}{1+\xi^2} \quad \text{——} \quad (2)$$

Solve (1) & (2) for either $F_c(y(x))(\xi)$ or $F_s(y(x))(\xi)$.

$$F_s(y(x))(\xi) = \sqrt{\frac{2}{\pi}}\frac{5\xi - \xi^3}{(\xi^2+1)^2(\xi^2+4)}$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{3\xi}{1+\xi^2} + \frac{\xi}{(1+\xi^2)^2} - \frac{2\xi}{\xi^2+4}\right)$$

Take inverse sine transform on both sides to get

$$\boxed{y(x) = 3e^{-x} - 2e^{-2x} + xe^{-x}}$$

Initial conditions are verified: $y(0) = 1$, $y'(0) = -3e^{-x} + 4e^{-2x} - xe^{-x} + e^{-x}\Big|_{x=0}$

$= -3+4+1 = 2$.

$$\sqrt{\frac{2}{\pi}}\int_0^\infty x e^{-x}\sin\xi x\, dx = \frac{\xi}{(1+\xi^2)^2}$$

$$3y'\left(\cancel{-xe^{-x}} + e^{-x}\right)3 + \frac{2(\cancel{xe^{-x}})}{2y}$$

$$+\cancel{xe^{-x}} - e^{-x} - e^{-x}$$

$$= e^{-x}y''$$

Linear Integral equations:

$$y(x) + \int_{-\infty}^{\infty} y(t)\, k(x,t)\, dt = f(x), \quad x \in (-\infty, \infty).$$

If $k(x,t) = k(x-t)$, then

$$\hat{y}(\xi) + \sqrt{2\pi}\,\hat{y}(\xi)\cdot\hat{k}(\xi) = \hat{f}(\xi).$$

$$\Rightarrow \hat{y}(\xi) = \frac{\hat{f}(\xi)}{1+\sqrt{2\pi}\,\hat{k}(\xi)} \Rightarrow y(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{1+\sqrt{2\pi}\,\hat{k}(\xi)} e^{i\xi x}\, d\xi$$

Example: Solve $\int_{-\infty}^{\infty} |x-t|^{-1/2} y(t)\, dt = f(x)$, $x \in (-\infty, \infty)$

Soln: By Applying Fourier transform, we get

$$\widehat{|x|^{-1/2}}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|^{-1/2} e^{-i\xi x}\, dx.$$

$$\widehat{|x|^{-1/2} * y(x)}(\xi) = \sqrt{2\pi}\, \widehat{|x|^{-1/2}}(\xi) \cdot \hat{y}(\xi) = \hat{f}(\xi)$$

$$f * g(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot \hat{g}(\xi)\, e^{i\xi x}\, d\xi$$

$$\Rightarrow \hat{y}(\xi) = \frac{\hat{f}(\xi)\, |\xi|^{1/2}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \, i|\xi| \hat{f}(\xi) \cdot -|\xi|^{-1/2} i$$

$$= -\frac{1}{\sqrt{2\pi}} (i \xi \hat{f})(i \operatorname{sgn}(\xi) |\xi|^{-1/2}),$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x^{-1/2} e^{-i\xi x}\, dx - \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x^{-1/2} e^{i\xi x}\, dx$$

$$= -\frac{\sqrt{2\pi}}{2\pi} \widehat{f'(x)}(\xi) \cdot \widehat{|x|^{-1/2} \operatorname{sgn}(x)}(\xi)$$

$$= |\xi|^{-1/2}$$

$$\Rightarrow \widehat{\int_{-\infty}^{\infty} f(t) g(x-t)\, dt}(\xi) = \widehat{f * g}(\xi) = \sqrt{2\pi}\, \hat{f}(\xi) \cdot \hat{g}(\xi)$$

$$\hat{y}(\xi) = -\frac{1}{2\pi} \widehat{f'(x) * (|x|^{-1/2} \operatorname{sgn}(x))}(\xi) \Rightarrow y(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} f'(t) \cdot |x-t|^{-1/2} \operatorname{sgn}(x-t)\, dt.$$

Integral equation

$$F_{s(c)}(y(x))(\xi) = \sqrt{\frac{2}{\pi}}\int_0^\infty y(x)\,\binom{\sin \xi x}{\cos \xi x}\,dx$$

$$k(x,\xi) = \cos \xi x, \quad \sin \xi x$$

Inverse transform solution

$$y(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_{s(c)}(y(x))(\xi)\,\binom{\sin \xi x}{\cos \xi x}\,d\xi$$

I.E for $f(x)$:

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\,dx$$

Inversion giving solution

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{i\xi x}\,d\xi$$

Digression: Evaluate $\displaystyle\int_0^\infty x^{n-1} e^{-i\xi x}\,dx, \quad \xi > 0$

Let $i\xi x = t$

$i\xi\, dx = dt$

$$\int_0^\infty x^{n-1} e^{-i\xi x}\,dx = \int_0^\infty \left(\frac{t}{i\xi}\right)^{n-1} e^{-t}\,\frac{dt}{i\xi}.$$

$$= \left(-\frac{i}{\xi}\right)^n \int_0^\infty t^{n-1} e^{-t}\,dt$$

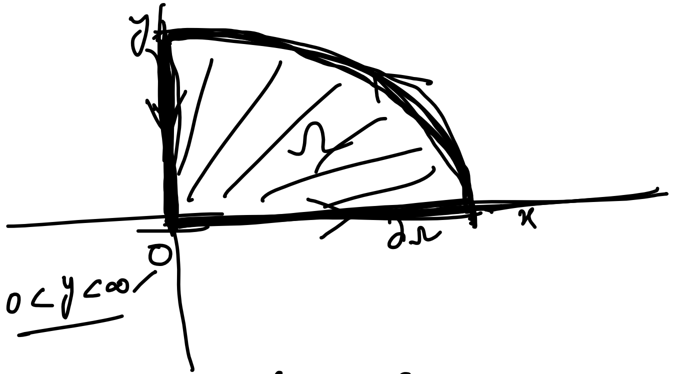
$$= \frac{\Gamma(n)}{\xi^n}\cdot\left(e^{-i\pi/2}\right)^n.$$

$$= \frac{\Gamma(n)}{\xi^n}\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right).$$

$n = 1/2, \quad \Gamma(1/2) = \sqrt{\pi}.$

$$\Gamma(n) := \int_0^\infty t^{n-1} e^{-t}\,dt$$

$(-i) = e^{-i\pi/2}$



$0 + iy, \quad 0 < y < \infty$

$$\int_{\partial\Omega} f(z)\,dz = \int_{\partial\Omega} z^{n-1} e^{i\xi z}\,dz = 0.$$

$z = re^{i\theta}, \quad 0 \le \theta \le \pi/2$

$dz = ire^{i\theta}\,d\theta.$

$$\left|\int_0^{\pi/2} r^{n-1} e^{in\theta} e^{i\xi r e^{i\theta}}\, ir e^{i\theta}\,d\theta\right|$$

As $r \to \infty$, $r^{n-1}\left(e^{i\xi r\cos\theta - \xi r\sin\theta}\right) \to 0$

$$\int_0^\infty x^{-1/2} e^{-i\xi x}\,dx = \frac{\sqrt{\pi}}{\xi^{1/2}}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) \checkmark$$

$$\Rightarrow \left.\begin{aligned} \int_0^\infty x^{-1/2} \cos\xi x\,dx &= \sqrt{\tfrac{\pi}{2}}\cdot \xi^{-1/2}, \quad \xi > 0 \\ \int_0^\infty x^{-1/2} \sin\xi x\,dx &= \sqrt{\tfrac{\pi}{2}}\; \xi^{-1/2}, \quad \xi > 0 \end{aligned}\right\} \checkmark$$

Example: Solve $y(x) - \frac{1}{2}\int_{-\infty}^{\infty} y(t)\, e^{-2|x-t|}\,dt = f(x)$, $-\infty < x < \infty$.

$$\int_{-\infty}^{\infty} e^{-2|x|} e^{-i\xi x}\,dx = \int_0^\infty e^{-2x} e^{-i\xi x}\,dx + \int_0^\infty e^{-2x} e^{i\xi x}\,dx$$

$$\hat{y}(\xi) - \frac{2}{\xi^2+4}\cdot\hat{y}(\xi) = \hat{f}(\xi)$$

$$\Rightarrow \hat{y}(\xi) = \hat{f}(\xi)\cdot\frac{\xi^2+4}{\xi^2+2} = \hat{f}(\xi) + \frac{2}{\xi^2+2}\hat{f}(\xi).$$

$$\Rightarrow \boxed{y(x) = f(x) + \frac{1}{\sqrt{2}}\left(e^{-\sqrt{2}|x|} * f(x)\right)}$$

Evaluate some integrals:

$$\widehat{f * g(x)}(\xi) = \sqrt{2\pi}\, \hat{f}(\xi) \cdot \hat{g}(\xi).$$

$$\Downarrow.$$

$$\int_{-\infty}^{\infty} f(-y)\, g(y)\, dy = \int_{-\infty}^{\infty} \hat{f}(\xi)\, \hat{g}(\xi)\, d\xi.$$

Example: Evaluate $I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}.$

$$I = \int_{-\infty}^{\infty} \frac{1}{(\xi^2+a^2)} \cdot \frac{1}{\underline{(\xi^2+b^2)}} \, d\xi$$

$$\Rightarrow \quad \hat{f}(\xi) = \frac{1}{\xi^2+a^2}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\xi^2+a^2} e^{i\xi x} \, d\xi.$$

$$g(x) = \frac{1}{2a} e^{-a|x|}, \quad \underline{a > 0}$$

$$\hat{g}(\xi) = \frac{1}{2a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{-i\xi x} \, dx$$

$$= \frac{1}{2a} \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax - i\xi x} \, dx + \frac{1}{\sqrt{2\pi}} \frac{1}{2a} \int_0^{\infty} e^{-ax + i\xi x} \, dx$$

$$\int_{\partial\Omega} f(z)\, dz = 2\pi i \operatorname*{Res}_{z=ia} f(z) = \underline{2\pi i \lim_{z \to ia} (z - ia) f(z)}$$

$$f(z) = \frac{e^{ixz}}{z^2+a^2}$$

$$\underline{z = \pm ia}$$

$$= \frac{1}{\sqrt{2\pi}\,2a} \frac{-1}{(a+i\xi)} e^{-(a+i\xi)x} \Big|_0^\infty - \frac{1}{2a\sqrt{2\pi}} \frac{e^{-(a-i\xi)x}}{a-i\xi} \Big|_0^\infty$$

$$= \frac{1}{2a\sqrt{2\pi}} \left[\frac{1}{a+i\xi} + \frac{1}{a-i\xi} \right] = \frac{1}{\sqrt{2\pi}} \frac{1 \cancel{2a}}{\cancel{2a}\; a^2+\xi^2}$$

$$\Rightarrow \quad \hat{g}(\xi) = \frac{1}{\sqrt{2\pi}\; a^2+\xi^2}$$

$$\widehat{\frac{\sqrt{2\pi}}{2a} e^{-a|x|}}(\xi) = \frac{1}{a^2+\xi^2}$$

$$I = \int_{-\infty}^{\infty} \widehat{\frac{\sqrt{2\pi}}{2a} e^{-a|x|}}(\xi) \; \widehat{\frac{\sqrt{2\pi}}{2b} e^{-b|x|}}(\xi) \, d\xi$$

$$= \int_{-\infty}^{\infty} \frac{\sqrt{2\pi}}{2a} e^{-a|x|} \cdot \frac{\sqrt{2\pi}}{2b} e^{-b|x|} \, dx \checkmark$$

$$= \frac{\pi}{2ab} \int_{-\infty}^{\infty} \underline{e^{-(a+b)|x|}} \, dx$$

$$= \frac{\pi}{\cancel{2}ab} \cancel{2} \cdot \int_{0}^{\infty} e^{-(a+b)x} \, dx$$

$$\underline{\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab} \cdot \frac{1}{a+b}.} \checkmark$$

$$\int_0^\infty F_c(f)(\xi)\, F_c(g)(\xi)\, d\xi = \int_0^\infty F_c(f)(\xi) \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos \xi x \, dx \, d\xi$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty F_c(f)(\xi)\, g(x) \cos \xi x \, d\xi \, dx$$

$$= \underline{\sqrt{\frac{2}{\pi}}} \int_0^\infty g(x) \underline{\int_0^\infty F_c(f)(\xi) \cos \xi x \, d\xi} \, dx$$

$$\int_0^\infty F_c(f)(\xi) \cdot F_c(g)(\xi) \, d\xi = \int_0^\infty g(x)\, f(x)\, dx .$$

IIIy, $\displaystyle\int_0^\infty F_s(f)(\xi) \cdot F_s(g)(\xi)\, d\xi = \int_0^\infty f(x)\, g(x)\, dx .$

Example: Evaluate $I=\displaystyle\int_0^\infty \frac{\sin ax}{x(a^2+x^2)}\,dx$, $a \in \mathbb{R}$.

$$I=\int_0^\infty \frac{1}{a^2+\xi^2}\,\frac{\sin a\xi}{\xi}\,d\xi$$

$$=\int_0^\infty F_c\left(\sqrt{\frac{\pi}{2}}\,\frac{e^{-ax}}{a}\right)(\xi)\, F_c\left(\sqrt{\frac{\pi}{2}}\,g(x)\right)(\xi)\,d\xi$$

$$=\frac{\pi}{2}\int_0^\infty \frac{e^{-ax}}{a}\cdot g(x)\,dx$$

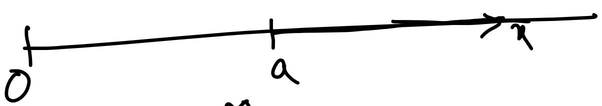
$$=\frac{\pi}{2a}\int_0^a e^{-ax}\,dx = -\frac{\pi}{2a}\,\frac{e^{-ax}}{a}\Big|_0^a$$

$$I = -\frac{\pi}{2a^2}\left(e^{-a^2}-1\right)=\left(1-e^{-a^2}\right)\frac{\pi}{2a^2}$$

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos\xi x\,dx = \sqrt{\frac{2}{\pi}}\cdot\frac{a}{\xi^2+a^2}$$

$$\Rightarrow\quad F_c\left(\sqrt{\frac{\pi}{2}}\,\frac{e^{-ax}}{a}\right)=\frac{1}{\xi^2+a^2}$$

$$g(x)=\begin{cases}1, & 0<x<a\\ 0, & x>a\end{cases}$$



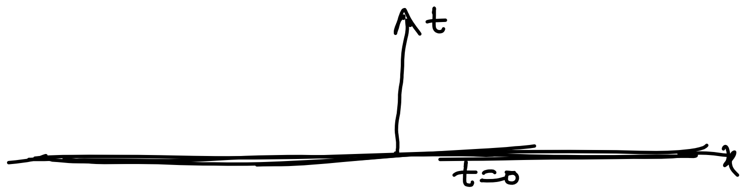
$$F_c(g)(\xi) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos\xi x\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^a \cos\xi x\,dx = \sqrt{\frac{2}{\pi}}\,\frac{\sin\xi x}{\xi}\Big|_0^a$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{\sin\xi a}{\xi}.$$

$$F_c\left(g(x)\sqrt{\frac{\pi}{2}}\right)=\frac{\sin\xi a}{\xi}.$$

D'Alembert's solution of wave equation by Fourier transform:



Initial value problem
$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R} \\ u_t(x,0) = g(x), & x \in \mathbb{R} \end{cases}$$

Soln: Apply Fourier transform, to get

$$\left.\begin{aligned} \frac{\partial^2}{\partial t^2} \hat{u}(\xi,t) &= c^2 (i\xi)^2 \hat{u}(\xi,t), \quad t > 0. \\ \hat{u}(\xi,0) &= \hat{f}(\xi) \ \& \ \frac{\partial \hat{u}(\xi,0)}{\partial t} = \hat{g}(\xi). \end{aligned}\right\}$$

$$\hat{u}_{tt} + c^2 \xi^2 \hat{u} = 0. \checkmark \qquad m^2 + c^2\xi^2 = 0$$

$$m = \pm c \xi i$$

$$\hat{u}(\xi,t) = c_1 \cos c\xi t + c_2 \sin c\xi t$$

$$\hat{u}(\xi,0) = c_1 = \hat{f}(\xi)$$

$$\hat{u}(\xi,t) = \hat{f}(\xi) \cos c\xi t + c_2 \sin c\xi t$$

$$\left.\frac{\partial \hat{u}(\xi,t)}{\partial t}\right|_{t=0} = \left. -\hat{f}(\xi)\, c\xi \sin c\xi t + c\xi c_2 \cos c\xi t \right|_{t=0} = \hat{g}(\xi)$$

$$\Rightarrow \quad c\xi c_2 = \hat{g}(\xi)$$

$$\Rightarrow \quad c_2 = \frac{\hat{g}(\xi)}{c\xi}.$$

$$\hat{u}(\xi,t) = \underline{\hat{f}(\xi) \cos c\xi t} + \frac{\hat{g}(\xi)}{c\xi} \cdot \sin c\xi t.$$

Take the inverse transform to get $u(x,t)$.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)\left(\frac{e^{ic\xi t}+e^{-ic\xi t}}{2}\right) e^{i\xi x}\,d\xi + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\hat{g}(\xi)}{c\xi}\left[\frac{e^{ic\xi t}-e^{-ic\xi t}}{2i}\right] e^{i\xi x}\,dx.$$

$$= \underline{\frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\xi)\,e^{i\xi(x+ct)}\,d\xi + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\xi)\,e^{i\xi(x-ct)}\,d\xi\right]} + \underline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{g}(\xi)}{c\xi}\left[\frac{e^{ic\xi t}-e^{-ic\xi t}}{2i}\right]e^{i\xi x}\,d\xi.}$$

$$= I_1 + I_2.$$

$$I_1 = \frac{1}{2}\cdot\left[f(x+ct) + f(x-ct)\right].$$

Let $\phi(x) = \int_{A}^{x} g(u)\,du$, then $\phi'(x) = g(x)$, $x \in (-\infty,\infty)$

$$\Rightarrow i\xi\,\hat{\phi}(\xi) = \hat{g}(\xi).$$

$$\frac{I}{2} = \frac{1}{2i}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\phi}(\xi)\left[e^{i\xi(x+ct)} - e^{i\xi(x-ct)}\right]d\xi\right]$$

$$= \frac{1}{2C}\left[\underline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\phi}(\xi)\,e^{i\xi(x+ct)}\,d\xi} - \underline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\phi}(\xi)\,e^{i\xi(x-ct)}}\,d\xi\right]$$

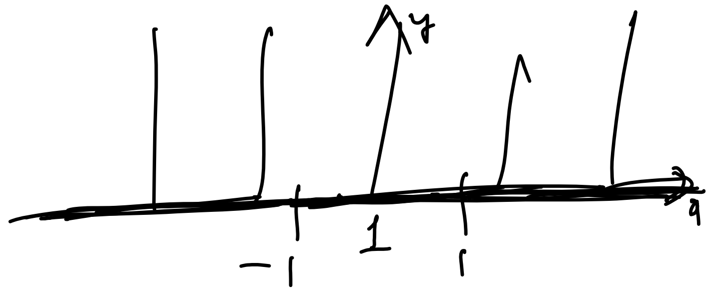
$t \geq 0$

$$= \frac{1}{2C}\left[\phi(x+ct) - \phi(x-ct)\right]$$

$$I_2 = \frac{1}{2C}\left[\int_{A}^{x+ct} g(u)\,du - \int_{A}^{x-ct} g(u)\,du\right] = \frac{1}{2C}\int_{x-ct}^{x+ct} g(u)\,du. \checkmark$$

$$\checkmark\ u(x,t) = \frac{1}{2}\Big(f(x+ct) + f(x-ct)\Big) + \frac{1}{2C}\int_{x-ct}^{x+ct} g(u)\,du, \qquad x \in \mathbb{R},\ t \geq 0$$

Solution of 2-dimensional Laplace equation:



$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad x \in \mathbb{R}, \quad y > 0.$$

$u_{xx} + u_{yy} = 0$

$$u(x,0) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases};$$

$u, \nabla u \to 0$ as $\sqrt{x^2+y^2} \to \infty$

$$(i\xi)^2 \hat{u}(\xi, y) + \frac{\partial^2}{\partial y^2} \hat{u}(\xi, y) = 0; \quad y > 0$$

$\hat{u}_{yy} - \xi^2 \hat{u} = 0$

$m^2 - \xi^2 = 0$

$m = \pm \xi, \quad \xi \in \mathbb{R}.$

$$\hat{u}(\xi, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)\, e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\xi x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-i\xi x}}{-i\xi} \Bigg|_{x=-1}^{x=1} = \frac{1}{\sqrt{2\pi}} \frac{\left[e^{-i\xi} - e^{i\xi}\right]}{-i\xi}$$

$$\hat{u}(\xi,0) = \frac{1}{\sqrt{2\pi}} \cdot \frac{\cancel{+}2\cancel{i}\sin\xi}{\cancel{+i}\,\xi} = \sqrt{\frac{2}{\pi}}\,\frac{\sin\xi}{\xi}.$$

$\hat{u}(\xi,y) \to 0$ as $x^2+y^2 \to \infty$.

$$\hat{u}(\xi,y) = C_1 e^{|\xi|y} + C_2 e^{-|\xi|y}. \qquad y>0$$

Since $\hat{u} \to 0$ as $x^2+y^2 \to \infty$, $C_1 = 0$

$$\Rightarrow \hat{u}(\xi,y) = C_2 e^{-|\xi|y}$$

$$\sqrt{\frac{2}{\pi}}\,\frac{\sin\xi}{\xi} = \hat{u}(\xi,0) \Rightarrow \sqrt{\frac{2}{\pi}}\,\frac{\sin\xi}{\xi} = C_2$$

$$\hat{u}(\xi,y) = \sqrt{\frac{2}{\pi}}\, e^{-|\xi|y}\,\frac{\sin\xi}{\xi}; \qquad y>0.$$

$$\hat{u}(\xi,0) = \sqrt{\frac{2}{\pi}}\,\frac{\sin\xi}{\xi}$$

$$\sqrt{\frac{\pi}{2}}\, u(x,0) = \mathcal{F}^{-1}\left(\frac{\sin\xi}{\xi}\right).$$

Apply inverse fourier transform to get $\underline{u(x,y)}$.

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-|\xi| y} \frac{\sin \xi}{\xi} \cdot e^{i\xi x} \, d\xi .$$

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-|\xi| y + i\xi x} \cdot \frac{\sin \xi}{\xi} \, d\xi ; \quad \underline{x \in \mathbb{R}, \ y > 0}$$

$$\check{f(\xi)}(x) = \underline{\mathcal{F}}^{-1}\big(f(\xi)\big)(x)$$

$$f * g(x) = \mathcal{F}^{-1}\left(\sqrt{2\pi}\, \hat{f}(\xi) \cdot \hat{g}(\xi)\right)(x)$$

If $f(x) = \check{f(\xi)}(x)$, then

$$\underline{\check{f} * g(x)} = \mathcal{F}^{-1}\left(\sqrt{2\pi}\; f(\xi) \cdot \hat{g}(\xi)\right)(x) .$$

$$u(x,y) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underline{e^{-|\xi|y}} \cdot \frac{\sin \xi}{\xi} e^{i\xi x} \, d\xi \, . = \sqrt{\frac{2}{\pi}\frac{1}{2\pi}} \underline{\mathcal{F}^{-1}\left(\sqrt{2\pi}\, e^{-|\xi|y} \cdot \frac{\sin\xi}{\xi}\right)}$$

$$u(x,y) = \frac{1}{\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} \underline{F^{-1}\left(\sqrt{2\pi}\, e^{-|\xi|y}\right)(z)} \quad \underline{F^{-1}\left(\frac{\sin\xi}{\xi}\right)(x-z)} \, dz \, .$$

$$\mathcal{F}^{-1}\left(\sqrt{2\pi}\, e^{-|\xi|y}\right)(x) = \cancel{\frac{1}{\sqrt{2\pi}}} \int_{-\infty}^{\infty} \cancel{\sqrt{2\pi}}\, e^{-|\xi|y} e^{i\xi x} \, d\xi \, , \quad \underline{y > 0}$$

$$= \int_{0}^{\infty} e^{-\xi y} e^{i\xi x} \, d\xi + \int_{-\infty}^{0} e^{\xi y} e^{i\xi x} \, d\xi .$$

$$= \int_{0}^{\infty} e^{-\xi(y - ix)} \, d\xi + \int_{-\infty}^{0} e^{\xi(y+ix)} \, d\xi$$

$$= \left. \frac{-e^{-\xi(y-ix)}}{y-ix} \right|_{\xi=0}^{\infty} + \left. \frac{e^{\xi(y+ix)}}{y+ix} \right|_{-\infty}^{0}.$$

$$= \frac{1}{y-ix} + \frac{1}{y+ix} = \frac{2y}{y^2+x^2}.$$

$$u(x,y) = \frac{1}{\pi \cancel{\sqrt{2\pi}}} \int_{-\infty}^{\infty} \frac{\cancel{2}y}{y^2+z^2} \cancel{\sqrt{\frac{\pi}{2}}} \cdot u(x-z,0)\, dz.$$

$$\boxed{u(x,y) = \frac{1}{\pi} \int_{-1+x}^{1+x} \frac{y}{y^2+z^2}\, dz}$$

$$|x-z| \le 1$$

$$-1 \le z-x \le 1$$

$$-1+x \le z \le 1+x$$

Solution of Heat equation

$$\frac{\partial u(x,t)}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0.$$

Initial Condition $\rightarrow$ $u(x,0) = f(x)$, $-\infty\ x < \infty$

Boundary Condition: $u(x,t) \rightarrow 0$ as $|x| \rightarrow \infty$.

Solution: Apply Fourier transform on 'x' variable to get

$$\frac{\partial \hat{u}(\xi,t)}{\partial t} = k\,(i\xi)^2\,\hat{u}(\xi,t), \quad \xi \in \mathbb{R},\ t > 0$$

$$\hat{u}(\xi,0) = \hat{f}(\xi), \quad \xi \in \mathbb{R}.$$

$$\frac{\partial \hat{u}(\xi,t)}{\partial t} + k\xi^2 \hat{u}(\xi,t) = 0, \quad t>0, \quad \xi \in (-\infty,\infty).$$

$$\hat{u}(\xi,0) = \hat{f}(\xi)$$

$$I.F = e^{\int k\xi^2 dt} = e^{k\xi^2 t}$$

$$\frac{dy}{dx} + py = 0$$

$$e^{\int p\,dx}\left(\frac{dy}{dx} + py\right) = 0$$

$$\frac{d}{dx}\left(e^{\int p\,dx} \cdot y\right) = 0$$

$$\hat{u}(\xi,t) = e^{-k\xi^2 t} C$$

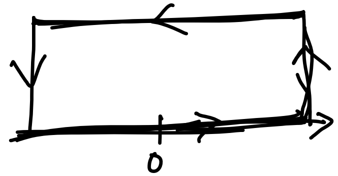
At $t=0$, $\hat{f}(\xi) = C$

$$\hat{u}(\xi,t) = \hat{f}(\xi)\, e^{-k\xi^2 t}, \quad \xi \in \mathbb{R},\ t>0$$

Taking inverse Fourier transform, we get

$$\boxed{u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{-k\xi^2 t}\, e^{i\xi x}\, d\xi, \quad x \in (-\infty,\infty),\ t>0}$$

$$\widehat{e^{-a^2x^2}}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2} e^{-i\xi x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(ax+\frac{i\xi}{2a}\right)^2} e^{-\frac{\xi^2}{4a^2}}\,dx .$$



$$ax + \frac{i\xi}{2a} = t$$

$$a\,dx = dt$$

$$= e^{-\frac{\xi^2}{4a^2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{-\left(ax+\frac{i\xi}{2a}\right)^2} dx .$$

$$= e^{-\frac{\xi^2}{4a^2}} \frac{1}{\sqrt{2\pi}\,a}\int_{-\infty}^{\infty} e^{-t^2}\,dt$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \pi$$

$$= e^{-\frac{\xi^2}{4a^2}} \frac{1}{\sqrt{2\pi}\;a}\sqrt{\pi}$$

$$= \frac{e^{-\frac{\xi^2}{4a^2}}}{\sqrt{2}\;a} .$$

$$\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2}\,a}\, e^{-\frac{\xi^2}{4a^2}}\right)(x) = e^{-a^2x^2}.$$

$$a^2 = \frac{1}{4kt}, \quad a = \frac{1}{2\sqrt{kt}}.$$

$$\sqrt{2kt}\cdot e^{-\xi^2 kt} = \widehat{e^{-\frac{x^2}{4kt}}}(\xi)$$

$$\hat{u}(\xi,t) = \hat{f}(\xi)\cdot e^{-k\xi^2 t}.$$

$$= \hat{f}(\xi)\cdot \frac{1}{\sqrt{2kt}}\, \widehat{e^{-\frac{x^2}{4kt}}}(\xi)$$

$$\hat{u}(\xi,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2kt}} \sqrt{2\pi}\ \hat{f}(\xi) \cdot \widehat{e^{-x^2/4kt}}(\xi).$$

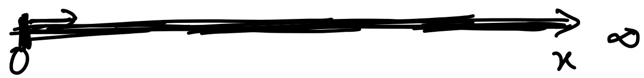
$$\widehat{f*g(x)}(\xi) = \sqrt{2\pi}\,\hat{f}(\xi) \cdot \hat{g}(\xi)$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2kt}} \cdot \int_{-\infty}^{\infty} f(y) \cdot e^{-\frac{(x-y)^2}{4kt}}\, dy.$$

$$\boxed{u(x,t) = \frac{1}{2\sqrt{k\pi t}} \cdot \int_{-\infty}^{\infty} f(y)\, e^{-\frac{(x-y)^2}{4kt}}\, dy, \quad x \in (-\infty,\infty),\ t > 0}$$

✓

* Solve $\dfrac{\partial u}{\partial t} = k\,\dfrac{\partial^2 u}{\partial x^2}$, $x \in (0, \infty)$, $t > 0$



$\underline{\text{I.C}}$: $u(x,0) = 0$, $x \in (0, \infty)$.

$\underline{\text{B.C's}}$: $\begin{cases} u(0,t) = C & \text{(Dirichlet boundary condition)} \checkmark \\ u(x,t),\ \underline{u_x(x,t) \to 0} & \text{as } x \to \infty. \end{cases}$

$\boxed{\underline{u_x(0,t)} \text{ is given as B.C's then } \downarrow \text{ (Neumann Boundary condition)}}$
$\downarrow$
$\underline{\text{Fourier Cosine transform.}}$

$\underline{\text{Solution:}}$ We apply Fourier Sine transform to 'x' variable.

$$\frac{\partial}{\partial t}\big(F_s(u)(\xi,t)\big) = k\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\,\sin \xi x\,dx.$$

$$= k\sqrt{\frac{2}{\pi}}\left[\cancel{\frac{\partial u}{\partial x}\,\underline{\sin \xi x}\Big|_{x=0}^{\infty}} - \xi\int_0^\infty \frac{\partial u}{\partial x}\cdot \cos \xi x\,dx\right]$$

$$= -k\xi\sqrt{\frac{2}{\pi}}\left[u \cos \xi x\Big|_{x=0}^{\infty} + \xi\int_0^\infty u(x,t)\,\sin \xi x\,dx\right]$$

$$= -k\xi^2 F_s(u)(\xi) + k\xi\sqrt{\frac{2}{\pi}}\,c.$$

$$\Rightarrow \quad \frac{\partial}{\partial t} F_s(u)(\xi,t) + k\xi^2 F_s(u)(\xi,t) = \sqrt{\frac{2}{\pi}}\,k\xi c. \checkmark$$

$$I.F = e^{\int k\xi^2\,dt} = e^{k\xi^2 t}.$$

$$\int \frac{\partial}{\partial t}\left(e^{k\xi^2 t} \cdot F_s(u)(\xi,t)\right)dt = \int \sqrt{\frac{2}{\pi}}\,ck\xi\, e^{k\xi^2 t}\,dt + c_1, \quad t > 0$$

where c_1 is an integration constant.

$$\cancel{e^{k\xi^2 t}}\, F_s(u)(\xi,t) = \sqrt{\frac{2}{\pi}}\,\frac{c\cancel{k}\xi\, \cancel{e^{k\xi^2 t}}}{\cancel{k}\xi^2} + c_1\, e^{-k\xi^2 t}$$

$$F_s(u)(\xi,t) = \sqrt{\frac{2}{\pi}}\,\frac{c}{\xi} + c_1\, e^{-k\xi^2 t} \checkmark$$

Since $\underline{u(x,0) = 0}$, we have

$$\underline{F_s(u)(\xi, 0) = 0}$$

$$0 = \sqrt{\frac{2}{\pi}}\frac{c}{\xi} + c_1 .$$

$$\Rightarrow c_1 = -\sqrt{\frac{2}{\pi}}\frac{c}{\xi} .$$

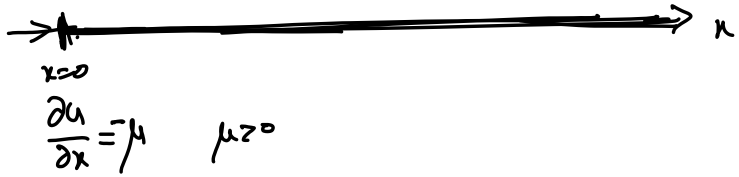
$$\Rightarrow \quad F_s(u)(\xi, t) = \sqrt{\frac{2}{\pi}}\frac{c}{\xi}\left(1 - e^{-k\xi^2 t}\right), \quad t > 0, \ \xi > 0.$$

Taking inverse transform, to get

$$\boxed{u(x,t) = \frac{2c}{\pi}\int_0^\infty \frac{\left(1 - e^{-k\xi^2 t}\right)}{\xi}\sin \xi x \, d\xi, \quad x > 0, \ t > 0.}$$

Remark: As $t \to \infty$, $\underline{u(x,t) = c, \ \forall\ x > 0}$. ✓

* Solve $\dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2}$, $x > 0$



I.C: $u(x,0) = 0, \ \forall\ x \in (0,\infty)$ ✓

B.C.s:
$$\begin{cases} \dfrac{\partial u(0,t)}{\partial x} = -\mu. \\ u(x,t),\ u_x(x,t) \longrightarrow 0 \text{ as } x \to \infty. \end{cases}$$

Solution: Apply Fourier Cosine transform to $u(x,t)$ w.r.to 'x' variable.

$$\frac{\partial}{\partial t} F_c(u)(\xi,t) = k \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos \xi x \, dx.$$

$$= k\sqrt{\frac{2}{\pi}}\left[\underline{\frac{\partial u}{\partial x}} \cos\xi x \Big|_{x=0}^{\infty} + \xi \int_0^{\infty} \frac{\partial u}{\partial x} \sin\xi x \, dx \right]$$

$$= k\sqrt{\frac{2}{\pi}}\,\mu + \xi k \sqrt{\frac{2}{\pi}} \left[\cancel{u(x,t) \sin\xi x \Big|_{x=0}^{\infty}} - \xi \int_0^{\infty} u(x,t) \cos\xi x \, dx \right]$$

$$= k\sqrt{\frac{2}{\pi}}\,\mu - \xi^2 k \, F_c(u)(\xi,t)$$

$$\Rightarrow \quad \frac{\partial}{\partial t} F_c(u)(\xi,t) + \underline{\xi^2 k} \, F_c(u)(\xi,t) = k\mu\sqrt{\frac{2}{\pi}}.$$

$$\int \frac{\partial}{\partial t}\left(\underline{e}^{\xi^2 k t} F_c(u)(\xi,t) \right) dt = \int k \mu \sqrt{\frac{2}{\pi}} \, e^{\xi^2 k t} \, dt + c, \quad t > 0. \quad \xi \geq 0$$

where c is integration constant

$$\cancel{e^{\xi^2 k t}} F_c(u)(\xi,t) = \cancel{k}\mu\sqrt{\frac{2}{\pi}} \frac{\cancel{e^{\xi^2 k t}}}{\cancel{k}\xi^2} + c \cdot e^{-\xi^2 k t}$$

$$\Rightarrow \quad F_c(u)(\xi,t) = \sqrt{\frac{2}{\pi}}\frac{\mu}{\xi^2} + c\, e^{-\xi^2 k t}.$$ ✓

Since $u(x,0)=0$, we get $F_c(u)(\xi,0)=0$:.

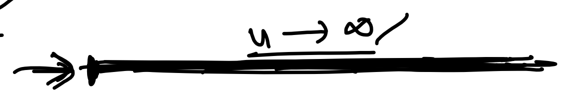
$$0 = \sqrt{\frac{2}{\pi}}\frac{\mu}{\xi^2} + c$$

$$\Rightarrow \quad F_c(u)(\xi,t) = \sqrt{\frac{2}{\pi}}\frac{\mu}{\xi^2}\left(1 - e^{-\xi^2 k t}\right), \quad \begin{matrix} t>0 \\ \xi \geq 0 \end{matrix}.$$

Taking inverse Fourier Cosine transform to get the solution

$$\boxed{u(x,t) = \frac{2\mu}{\pi}\int_0^\infty \frac{\left(1 - e^{-\xi^2 k t}\right)}{\xi^2}\cos \xi x \, d\xi, \quad \begin{matrix} x>0 \\ t>0 \end{matrix}}$$ ✓

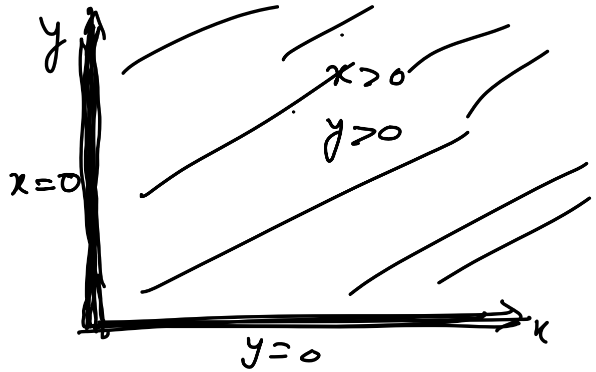
Remark: As $t \to \infty$, $\quad u(x,t) = \dfrac{2}{\pi}\mu \cdot \displaystyle\int_0^\infty \frac{\cos \xi x}{\xi^2}\, d\xi \quad \forall\, x > 0$



* Solve $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0; \quad 0 < x < \infty, \quad 0 < y < \infty$

$u(0,y) = c, \quad u(x,0) = 0.$

$u, \nabla u \to 0$ as $x^2 + y^2 \to \infty$.



Solution: Use Fourier sine transform, to get

$$\frac{\partial^2}{\partial y^2} F_s(u)(\xi, y) + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cdot \sin \xi x \, dx = 0$$

$$\frac{\partial^2}{\partial y^2} F_s(u)(\xi, y) + \sqrt{\frac{2}{\pi}} \left[\cancel{\frac{\partial u(x,y)}{\partial x} \sin \xi x \Big|_{x=0}^{\infty}} - \xi \int_0^{\infty} \frac{\partial u}{\partial x} \cos \xi x \, dx \right] = 0$$

$$\Rightarrow \frac{\partial^2 F_s(u)(\xi, y)}{\partial y^2} - \sqrt{\frac{2}{\pi}} \xi \left[\underline{u(x,y)} \cos \xi x \Big|_{x=0}^{\infty} + \xi \int_0^{\infty} \underline{u(x,y) \sin \xi x \, dx} \right] = 0$$

e^{yk}

$k^2 - \xi^2 = 0$

$k = \pm \xi$

$$\Rightarrow \underline{\frac{\partial^2}{\partial y^2} F_s(u)(\xi, y) - \xi^2 F_s(u)(\xi, y) = -\sqrt{\frac{2}{\pi}} \xi C.}$$

If $u(x, y) \to 0$, as $y \to \infty$,

$\underline{F_s(u)(\xi, y) \to 0 \text{ as } y \to \infty}$

$$F_s(u)(\xi, y) = \cancel{C_1 e^{\xi y}} + C_2 e^{-\xi y} + \sqrt{\frac{2}{\pi}} \frac{C}{\xi}; \quad \xi > 0, \; y > 0.$$

C_1, C_2 are constants.

$$\Rightarrow F_s(u)(\xi, y) = C_2 e^{-\xi y} + \sqrt{\frac{2}{\pi}} \frac{C}{\xi}, \quad \xi > 0, \; y > 0$$

Since $\underline{u(x,0)=0}$, we have $F_s(u)(\xi,0)=0$. ✓

$$0 = C_2 + \sqrt{\frac{2}{\pi}}\,\frac{C}{\xi}.$$

$$\Rightarrow \quad F_s(u)(\xi,y) = \sqrt{\frac{2}{\pi}}\,\frac{C}{\xi}\left(1 - e^{-\xi y}\right)$$

Inversion gives,

$$u(x,y) = \frac{2C}{\pi}\int_0^\infty \frac{1-e^{-\xi y}}{\xi}\sin \xi x \, d\xi, \quad \begin{matrix}\xi>0\\ x>0\end{matrix}$$

$$u(x,y) = C - \frac{2C}{\pi}\int_0^\infty e^{-\xi y}\frac{\sin \xi x}{\xi}\, d\xi; \quad \begin{matrix}\xi>0\\ x>0\end{matrix}$$ ✓

Since $\displaystyle\int_0^\infty e^{-\xi y}\sin \xi x \, d\xi = \frac{x}{x^2+y^2}, \quad y>0$

Integrate w.r.to 'y' from y to ∞, to get

$$\int_0^\infty \int_y^\infty e^{-\xi y}\, dy \sin \xi x \, d\xi = x\int_y^\infty \frac{1}{x^2+y^2}\, dy.$$

$$\int_0^\infty \left. \frac{-e^{-\xi y}}{\xi} \right|_{y=y}^{y=\infty} \sin \xi x \, d\xi = \int_y^\infty \frac{1}{1+(y/x)^2}\, d(y/x) = \tan^{-1}(y/x)\Big|_{y=y}^{\infty}$$

$$\int_0^\infty e^{-\xi y} \frac{\sin \xi x}{\xi}\, d\xi = \underline{\frac{\pi}{2} - \tan^{-1}(y/x)}.$$

$$= \tan^{-1}(x/y) \checkmark$$

$$\boxed{u(x,y) = C - \frac{2C}{\pi} \tan^{-1}(x/y) \; ; \quad \begin{array}{l} x > 0 \\ y > 0 \end{array}} \checkmark$$