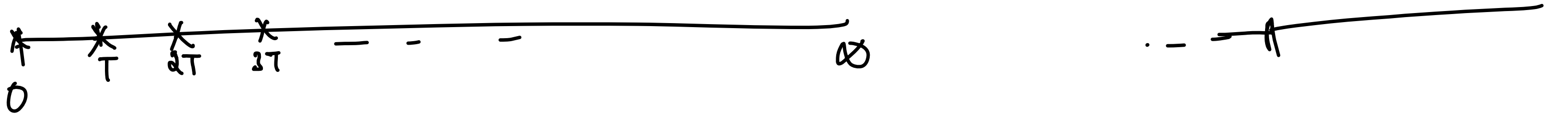
Z - transforms

$f(t)$, 

discrete sample $\longrightarrow$ $f(0),\ f(T),\ f(2T),\ - - - -$

<u>sample function</u> $\quad f^*(t) = \sum_{n=0}^{\infty} f(nT)\ \delta(t-nT), \qquad t \in (0, \infty)$

$$\int_0^{\infty} f^*(t)\ dt = f(0) + f(T) + f(2T) + - - - -$$

$$\mathcal{L}\left(f^*(t)\right)(s) = \int_0^\infty f^*(t)\, e^{-st}\, dt$$

$$= \int_0^\infty \sum_{n=0}^\infty f(nT)\, \delta(t-nT)\, e^{-st}\, dt$$

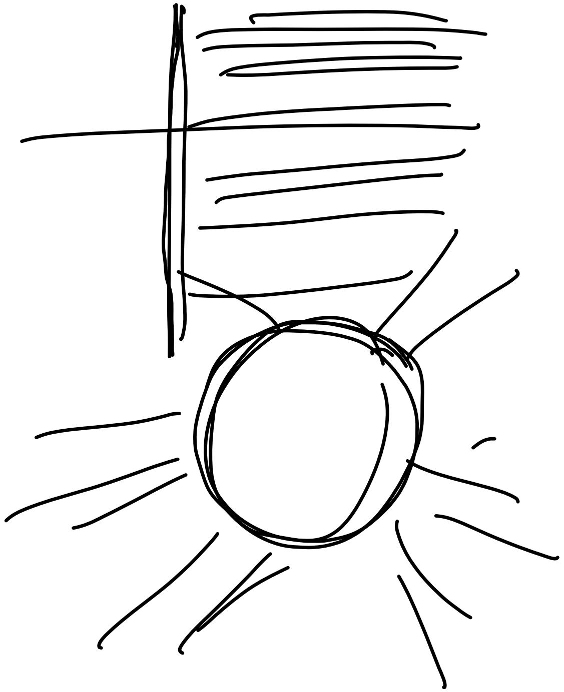
$$= \sum_{n=0}^\infty f(nT) \int_0^\infty e^{-st}\, \delta(t-nT)\, dt$$

$$\mathcal{L}(f^*(t))(s) = \sum_{n=0}^\infty f(nT) \cdot e^{-snT},$$

$s = \alpha + i\beta$

$\alpha = 0$.

$Re(s) = \alpha > 0$



Let $z = e^{sT}$ then

$$Z\left(f^*(t)\right)(z) := \mathcal{L}\left(f^*(t)\right)(z) := \sum_{n=0}^{\infty} f(nT)\, z^{-n}, \qquad |z| > e^{Re(s)T}.$$

$f^*(n) + \sin n\pi = f^*(n)$

This is called z-transform of the sample function $f^*(t)$.

or the sample $\{f(nT)\}_{n=0}^{\infty}$.

z-transform is unique

$\{f(n)\} \xrightarrow{1-1} Z(f(n))(z)$

$|e^{-\alpha}|\ |e^{-i\beta}| = 1,$

$T = 1.$

<u>examples</u>: 1. If $\{f(n)\}_{n=0}^{\infty} = 1$,

$$Z(f(n))(z) = \sum_{n=0}^{\infty} z^{-n}, \qquad |z| > 1$$

$$Z\left(f^*(t) + \sin \pi t\right)(z) = Z\left(f^*(t)\right)(z)$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$= \frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1}, \qquad |z| > 1.$$

2. If $f(n) = a^n$, $a \neq 0$,

$$Z(f(n))(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^{-n}, \quad |z| > |a|.$$

$$= \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}, \quad |z| > |a|.$$

3. If $f(n) = n$ then

$$Z(f(n))(z) = \sum_{n=0}^{\infty} n z^{-n} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$$

$$= -z \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^{-n}\right), \quad |z| > 1$$

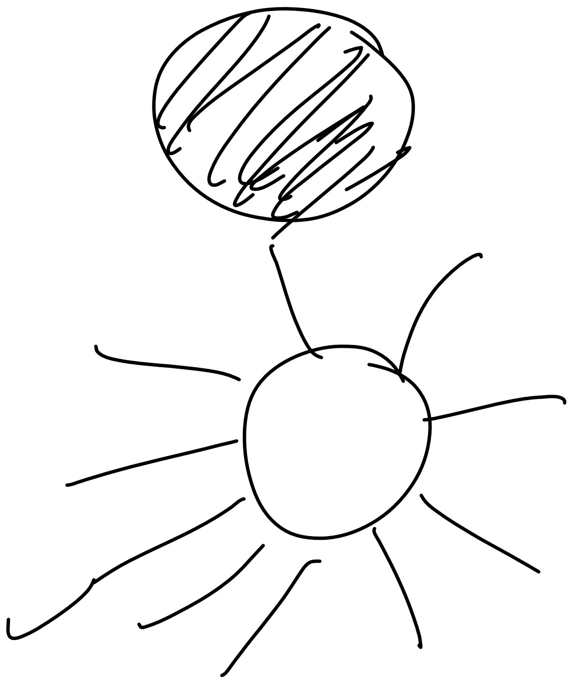
$$= -z \frac{d}{dz}\left(\frac{z}{z-1}\right), \quad |z| > 1$$

$$= -z \cdot \left(\frac{1}{z-1} - \frac{z}{(z-1)^2}\right)$$

$$= -\frac{z}{z-1} + \frac{z^2}{(z-1)^2}$$

$$= \frac{-z(z-1) + z^2}{(z-1)^2}$$

$$\boxed{Z(n)(z) = \frac{z}{(z-1)^2}, \quad |z| > 1}$$



4. If $f(n) = e^{inx}$. Then

$$Z(f(n))(z) = \sum_{n=0}^{\infty} e^{inx} \cdot z^{-n}, \quad |z| > 1$$

$$= \sum_{n=0}^{\infty} \left(\frac{z}{e^{ix}}\right)^{-n}, \quad |z| > 1 \qquad \text{since } \left|\frac{z}{e^{ix}}\right| = |z|$$

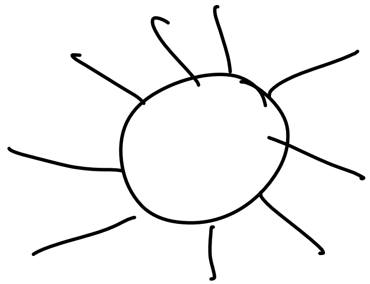
$$= \frac{z}{z - e^{ix}}, \quad |z| > 1 \ ✓$$

get $\underline{Z(\cos nx)(z)} = \sum_{n=0}^{\infty} \left(\frac{e^{inx} + e^{-inx}}{2}\right) z^{-n}$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{e^{ix}}\right)^{-n} + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{e^{-ix}}\right)^{-n} \ ✓$$

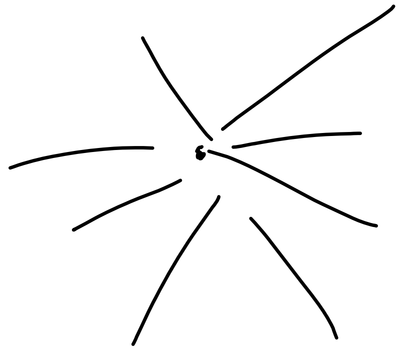
$$= \frac{1}{2} \cdot \frac{z}{z - e^{ix}} + \frac{1}{2} \cdot \frac{z}{z - e^{-ix}}; \quad |z| > 1$$

$$= \frac{z}{\cancel{2}} \, \frac{\cancel{2}z - \cancel{2}\cos x}{z^2 - 2z\cos x + 1} = \frac{z(z-\cos x)}{z^2 - 2z\cos x + 1}, \quad |z| > 1$$



$$Z(\sin nx)(z) = \frac{z \sin x}{z^2 - 2z\cos x + 1}, \quad |z| > 1$$

5. If $f(n) = \frac{1}{n!}$, Then



$$Z(f(n))(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = e^{1/z}, \quad \forall z \neq 0.$$

6. $$Z(n^2)(z) = \sum_{n=0}^{\infty} \underline{n^2 z^{-n}},$$

$$= -\sum_{n=0}^{\infty} z \frac{d}{dz}\left(n z^{-n}\right)$$

$$= -z\frac{d}{dz}\sum_{n=0}^{\infty} n z^{-n}$$

$$-n^2 z^{-n-1} = \frac{d}{dz}\left(n z^{-n}\right)$$

$$n^2 z^{-n} = -z\frac{d}{dz}\left(n z^{-n}\right).$$

$$= -z \cdot \frac{d}{dz}\left(\frac{z}{(z-1)^2}\right), \quad |z|>1.$$

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right), \quad |z|<1$$

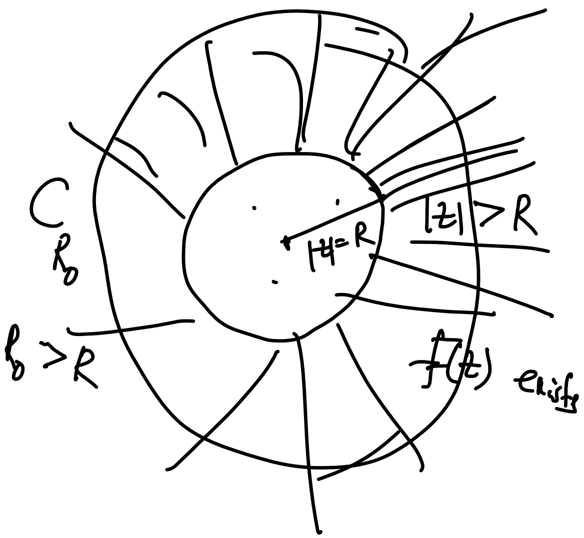
$$= \sum_{n=0}^{\infty} n z^{n-1}, \quad |z|<1$$

$$= -z \cdot \left[\frac{1}{(z-1)^2} - \frac{2z}{(z-1)^3}\right]$$

$$= -z\left[\frac{z-1-2z}{(z-1)^3}\right]$$

$$Z(n^2)(z) = -z\left[\frac{-z-1}{(z-1)^3}\right] = \frac{z(z+1)}{(z-1)^3}, \quad |z|>1.$$

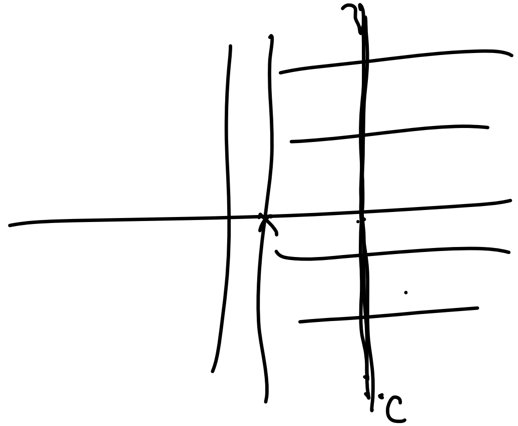
$$\{f(n)\}_{n=0}^{\infty} \longrightarrow Z(f(n))(z) =: F(z)$$



If I get $\underline{f}(n) = Z^{-1}(F(z))(n)$, $n = 0, 1, 2, \ldots$

<u>Inversion</u>:

$$\underline{f}(n) = \frac{1}{2\pi i}\int_{|z|=R_0} F(z)\, z^{n-1}\, dz, \quad R_0 > R.$$



Where $F(z)$ is <u>analytic</u> in $\underline{|z|>R}$.

<u>Proof</u>:

$$F(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n} = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots, \quad \underline{|z|>R}$$

$$\underline{\frac{1}{2\pi i}\int_{|z|=R_0} F(z)\, z^{n-1}\, dz} = \frac{1}{2\pi i}\int_{|z|=R_0} \sum_{n=0}^{\infty} f(n)\, z^{-n}\, z^{n-1}\, dz.$$

$$= \frac{1}{2\pi i}\left[\int_{|z|=R_0} f(0)\, z^{n-1}\, dz + \int_{|z|=R_0} f(1)\, z^{n-2}\, dz + \cdots + \underline{\int_{|z|=R_0} f(n)\, z^{-1}\, dz} + \int_{|z|=R_0} f(n+1)\, z^{-2}\, dz + \cdots\right]$$

$$z = R_0 e^{i\theta},\quad 0 \le \theta \le 2\pi$$

$$= \frac{1}{2\pi i}\cdot\int_{|z|=R_0} \underline{f(n)}\, \frac{dz}{z} = \frac{1}{\cancel{2\pi i}}\cdot f(n)\, \cancel{2\pi i} = \underline{f(n)}$$

$$\Rightarrow \quad Z^{-1}\big(F(z)\big)(n) = f(n) = \frac{1}{2\pi i}\int_{|z|=R_0} F(z)\, z^{n-1}\, dz.$$

Properties of Z-transform:

1. If $Z(f(n))(z) = F(z)$, then

$$Z(f(n-m)) = z^{-m} F(z)$$

$$Z(f(n+m)) = z^{m}\left[F(z) - \sum_{n=0}^{m-1} f(n) z^{-n}\right]$$

Proof:

$$Z(f(n-m))(z) = \sum_{n=0}^{\infty} f(n-m) z^{-n}$$

$n - m = r.$

$n = r + m$

$$= z^{-m}\sum_{r=0}^{\infty} f(r) z^{-r}.$$

$$= z^{-m} F(z)$$

$$Z\left(f(n+m)\right)(z) = \sum_{n=0}^{\infty} f(n+m)\, z^{-n} , \qquad n+m=r$$

$$n = r-m$$

$$= z^{m} \sum_{r=m}^{\infty} f(r)\, z^{-r}$$

$$= z^{m}\left[\sum_{r=0}^{\infty} f(r)\, z^{-r} - \sum_{r=0}^{m-1} f(r)\, z^{-r}\right]$$

$$= z^{m}\left[F(z) - \sum_{n=0}^{m-1} f(n)\, z^{-n}\right] \checkmark$$

2. If $Z(f(n))(z) = F(z)$, $|z|>R$, then $\quad |z|>R$

$$Z\left(a^{n} f(n)\right)(z) = F\left(\frac{z}{a}\right), \qquad |z|>|a|R \checkmark$$

$$Z(n^k f(n))(z) = (-1)^k \left(z\frac{d}{dz}\right)\left(z\frac{d}{dz}\right) \cdots \text{k times} \left(z\frac{d}{dz}\right) F(z), \quad |z| > R$$

$k = 0, 1, 2, \cdots$

Proof:

$$Z(a^n f(n))(z) = \sum_{n=0}^{\infty} a^n f(n) z^{-n}.$$

$$= \sum_{n=0}^{\infty} f(n) \cdot \left(\frac{z}{a}\right)^{-n}$$

$$= F\left(\frac{z}{a}\right), \quad |z| > |a| R$$

$$Z(n f(n))(z) = \sum_{n=0}^{\infty} n f(n) z^{-n}.$$

$$= z \sum_{n=0}^{\infty} n f(n) z^{-(n+1)}$$

$$= z \sum_{n=0}^{\infty} f(n) \cdot \left(- \frac{d}{dz}\left(z^{-n}\right)\right)$$

$$= -z \frac{d}{dz} \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$\mathcal{Z}(n f(n))(z) = -z \frac{d}{dz}\left(F(z)\right) \checkmark$$

$\underline{k=2}$:

$$\mathcal{Z}\left(n^2 f(n)\right)(z) = \mathcal{Z}\left(n \cdot \left(n f(n)\right)\right)(z) = -z \frac{d}{dz}\left(\underline{\mathcal{Z}(n f(n))}(z)\right)$$

$$= -z \frac{d}{dz}\left(-z \frac{d}{dz}\left(F(z)\right)\right)$$

$$= (-1)^2 \left(z \frac{d}{dz}\right)\left(z \frac{d}{dz}\right)\left(F(z)\right) . \checkmark$$

3. Convolution of two samples $f(n)$ and $g(n)$; $n = 0, 1, 2, \dots$

$$f * g(n) := \sum_{m=0}^{\infty} f(n-m)\, g(m)$$

$$Z\left(f * g(n)\right)(z) = Z(f(n))(z) \cdot Z(g(n))(z).$$

Proof: Since $f(n) = 0 = g(n)$, if $\underline{n < 0}$.

$$f * g(n) = \sum_{m=-\infty}^{\infty} f(n-m)\, g(m).$$ Then $f * g(n) = 0$, if $\underline{n < 0}$.

$$Z(f * g(n))(z) = \sum_{n=0}^{\infty} f * g(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} \underline{f * g(n)}\, z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(\underline{n-m})\, g(m)\, z^{-n}.$$

$$= \sum_{m=-\infty}^{\infty} g(m) \sum_{n=-\infty}^{\infty} f(n-m)\, z^{-n}.$$

$$= \sum_{m=-\infty}^{\infty} g(m)\, z^{-m} \sum_{\underbrace{n}_{r}=-\infty}^{\infty} f(\underbrace{n-m}_{r})\, z^{-(\underbrace{n-m}_{r})}$$

$n-\underline{m} = r.$

$$= \sum_{n=-\infty}^{\infty} g(n)\, z^{-n} \cdot \sum_{n=-\infty}^{\infty} f(n)\, z^{-n}$$

$$= \mathcal{Z}\big(f(n)\big)(z) \cdot \mathcal{Z}\big(g(n)\big)(z)\,; \qquad z \in D(F(z)) \cap D(G(z))$$

4. (initial value theorem)

If $Z(f(n))(z) = F(z)$, then

$f(0) = \lim_{z \to \infty} F(z)$. If $f(0)=0$, then $f(1) = \lim_{z\to\infty} z F(z)$.

Proof: $F(z) = \sum_{n=0}^{\infty} f(n) z^{-n} = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots$

$\lim_{z\to\infty} F(z) = f(0)$. ✓

If $f(0)=0$, $F(z) = \frac{f(1)}{z} + \frac{f(2)}{z^2} + \frac{f(3)}{z^3} + \cdots$

$\Rightarrow \lim_{z\to\infty} z F(z) = f(1)$.

5. (Final value Theorem)

If $Z(f(n))(z) = F(z)$, then

$$f(\infty) = \lim_{n\to\infty} f(n) = \lim_{z\to 1}\left((z-1)\,F(z)\right).$$

Proof:

$$Z\left(f(n+1) - f(n)\right)(z) = Z(f(n+1))(z) - Z(f(n))(z)$$

$$= z\cdot\left[F(z) - f(0)\right] - F(z).$$

$$= (z-1)\,F(z) - z f(0).$$

$$\Rightarrow \sum_{n=0}^{\infty}\left(f(n+1) - f(n)\right) z^{-n} = (z-1)\,F(z) - z f(0).$$

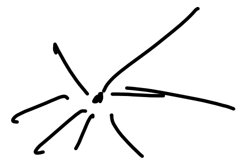
$$\Rightarrow \lim_{z\to 1} \lim_{m\to\infty} \sum_{n=0}^{m} \left(f(n+1) - f(n)\right) z^{-n} = \lim_{z\to 1}\left[(z-1)\,F(z) - z\,f(0)\right].$$

$$\lim_{m\to\infty} \sum_{n=0}^{m} \left(f(n+1) - f(n)\right) = \lim_{z\to 1} (z-1)\,F(z) - f(0).$$

$$\cancel{f(1)} - f(0)$$
$$+ \cancel{f(2)} - \cancel{f(1)}$$
$$+ \cancel{f(m)} - \cancel{f(m-1)}$$
$$+ f(m+1) - \cancel{f(m)}$$

$$\Rightarrow \lim_{m\to\infty} f(m+1) - \cancel{f(0)} = \lim_{z\to 1} (z-1)\,F(z) - \cancel{f(0)}.$$

$$\Rightarrow \boxed{f(\infty) = \lim_{n\to\infty} f(n) = \lim_{z\to 1} (z-1)\,F(z)} \checkmark$$



$$\left\{\frac{1}{n!}\right\}_{n\geq 0}^{\infty} = e^{1/z}$$

Inverse Z-transforms:

1. $Z^{-1}\left(e^{1/z}\right)$.

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$\Rightarrow \quad Z^{-1}\left(e^{1/z}\right)(n) = \frac{1}{n!}, \quad n = 0, 1, 2, 3, \cdots$$

$$Z(a^n) = \frac{z}{z-a}, \quad \underline{|z| > a}$$

$$\left|\frac{a}{z}\right| < 1$$

$$\Rightarrow \underline{|z| > a}$$

2\. Find $Z^{-1}\left(\frac{z}{z-a}\right)(n)$, where $\underline{|z| > |a|}$.

$$\frac{z}{z-a} = \frac{\cancel{z}}{\cancel{z}\underline{\left(1-\frac{a}{z}\right)}} = \left(1 - \frac{a}{z}\right)^{-1} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots$$

$$= \underline{\sum_{n=0}^{\infty} a^n z^{-n}}$$

$$\Rightarrow \quad Z^{-1}\left(\frac{z}{z-a}\right)(n) = a^n, \quad n = 0, 1, 2, 3, \cdots$$

3. $Z^{-1}(F(z))(n)$, when $F(z) = \dfrac{z}{z^2-6z+8}$.

$$F(z) = \frac{z}{(z-2)(z-4)} = \frac{1}{2}\left(\frac{z}{z-4} - \frac{z}{z-2}\right).$$

$$Z^{-1}(F(z))(n) = Z^{-1}\left(\frac{1}{2}\frac{z}{z-4}\right) - Z^{-1}\left(\frac{1}{2}\frac{z}{z-2}\right)$$

$$= \frac{1}{2}\left(4^n - 2^n\right), \quad n = 0, 1, 2, 3, \cdots$$

4. $Z^{-1}\left(\dfrac{z^2}{(z-a)(z-b)}\right)$.

$$\frac{z^2}{(z-a)(z-b)} = \frac{z}{(z-a)} \cdot \frac{z}{(z-b)}$$

$$= Z(a^n)(z) \cdot Z(b^n)(z)$$

$$= Z(a^n * b^n)(z)$$

$f(n) = a^n, \quad n > 0$

$f(n) = a^n = 0, \quad n < 0.$

$$\Rightarrow Z^{-1}\left(\frac{z^2}{(z-a)(z-b)}\right)(n) = Z^{-1}\left(Z(a^n * b^n)(z)\right)(n)$$

$$= a^n * b^n$$

$$= \sum_{m=0}^{\infty} a^{n-m} b^m$$

$$= \sum_{m=0}^{n} a^{n-m} b^m = a^n \sum_{m=0}^{n} \left(\frac{b}{a}\right)^m$$

$m = n+1$

$a^{n-m} = a^{-1} = 0$

$m = n+2, \; a^{n-m} = a^{-2} = 0$

$$= a^n \frac{1-\left(\frac{b}{a}\right)^{n+1}}{1-b/a} = \frac{a^{n+1}-b^{n+1}}{a-b}$$ ✓

5. $\mathcal{Z}^{-1}(F(z))$, where $F(z) = \dfrac{3z^2-z}{(z-1)(z-2)^2}$.

$$F(z) = \frac{2z}{z-1} - \frac{2z}{z-2} + \frac{5z}{(z-2)^2}$$

$$\mathcal{Z}^{-1}(F(z))(n) = 2\mathcal{Z}^{-1}\left(\frac{z}{z-1}\right) - 2\mathcal{Z}^{-1}\left(\frac{z}{z-2}\right) + \mathcal{Z}^{-1}\left(\frac{5z}{(z-2)^2}\right)$$

$$= 2 - 2\,2^n + \frac{5}{2}\cdot \mathcal{Z}^{-1}\left(\frac{2z}{(z-2)^2}\right).$$

$$= 2 - 2^{n+1} + 5\cdot n\cdot 2^{n-1}, \quad n = 0, 1, 2, \cdots$$

$$\mathcal{Z}(\underline{n}\,2^n) = -z\frac{d}{dz}\left(\frac{z}{z-2}\right)$$

$$= -z\left(\frac{1}{z-2} - z\cdot\frac{1}{(z-2)^2}\right)$$

$$= \frac{-z(\cancel{z}-2-\cancel{z})}{(z-2)^2}$$

$$= \frac{2z}{(z-2)^2}.$$

6. $$Z^{-1}\left(\frac{z(z+1)}{(z-1)^3}\right)$$

$$Z(H(n)) = Z(1) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1} \checkmark$$

$$H(n) = \{1, 1, 1, -----\}$$

$$H(n-1) = \{0, 1, 1, -----\}$$

$$Z(H(n-1))(z) = \sum_{n=1}^{\infty} z^{-n}$$

$$= \frac{1}{z}\cdot\sum_{m=0}^{\infty} z^{-m}$$

$$= \frac{1}{z}\cdot \underline{Z(H(n))}$$

$$= \frac{1}{\cancel{z}}\cdot\frac{\cancel{z}}{z-1} = \frac{1}{z-1}$$

$$Z^{-1}\left(\frac{z(z+1)}{(z-1)^3}\right)(n) = \left[Z^{-1}\left(\frac{z}{(z-1)^2}\cdot\frac{z}{(z-1)}\right) + Z^{-1}\left(\frac{z}{\underline{(z-1)^2}}\cdot\frac{1}{\underline{z-1}}\right)\right]$$

$$= \left[n * H(n) + n * H(n-1)\right]$$

$$= \left[n * \left[H(n) + H(n-1)\right]\right]$$

$$= \sum_{m=0}^{n}(n-m)\left(H(m) + H(m-1)\right) = \sum_{m=0}^{n} m\left[H(n-m) + H(n-m+1)\right].$$

$$= \begin{cases} 0, & n=0 \\ 1, & n=1 \\ 4, & n=2 \\ 9, & n=3 \end{cases}$$

$2+2-$

$$= n^2.$$

$$Z(n\cdot 1) = -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= -z\cdot\left(\frac{1}{z-1} - \frac{z}{(z-1)^2}\right)$$

$$= -z\left(\frac{\cancel{z}-1-\cancel{z}}{(z-1)^2}\right)$$

$$= \frac{z}{(z-1)^2}.$$

Applications of Z-transforms:

1. Solving difference equations.

I[st] order

* Solve.

$$f(n+1) - f(n) = 1, \quad n = 0, 1, 2, 3, \ldots$$

$$f(0) = 0$$

Soln: Apply Z-transform to the equation, we get

$$Z(f(n+1)) - Z(f(n)) = Z(\{1\})$$

$$z\left[Z(f(n)) - \sum_{r=0}^{0} f(r) z^{-r}\right] - Z(f(n)) = \frac{z}{z-1}, \quad |z| > 1$$

$$\Rightarrow (z-1)\, Z(f(n))(z) = \frac{z}{z-1}, \quad |z| > 1.$$

$$\Rightarrow \quad Z(f(n))(z) = \frac{z}{(z-1)^2}, \quad |z| > 1$$

Inversion gives

$$f(n) = n, \quad \forall\, n = 0, 1, 2, \ldots$$

$$Z\{n\} = -\frac{d}{dz} Z(\{1\})$$
$$= -\frac{d}{dz}\left(\frac{z}{z-1}\right)$$
$$= \frac{z}{(z-1)^2}$$

* solve $f(n+1) + 2f(n) = n, \quad n = 0, 1, 2, \dots$

$f(0) = 1$

Soln: Z - transform takes the equation to the form

$$z\left[F(z) - 1\right] + 2F(z) = \frac{z}{(z-1)^2}, \quad |z| > 1.$$

$$\Rightarrow (z+2)F(z) = z + \frac{z}{(z-1)^2}$$

$$\Rightarrow \quad F(z) = \frac{z}{z+2} + \frac{z}{(z+2)(z-1)^2}$$

$$F(z) = \frac{z}{z+2} + \frac{1}{9}\cdot\frac{z}{z+2} - \frac{1}{9}\cdot\frac{z}{z-1} + \frac{3}{9}\frac{z}{(z-1)^2}.$$

Inversion gives $\quad f(n) = (-2)^n + \frac{1}{9}(-2)^n - \frac{1}{9} + \frac{3}{9}n.$

$$\Rightarrow \quad f(n) = (-2)^n \frac{10}{9} + \frac{3}{9}n - \frac{1}{9}.$$

$$f(n) = \frac{1}{9}\left(3n - 1 + 10(-2)^n\right), \quad n = 0, 1, 2, \ldots$$

* $1, 1, 2, 3, 5, \ldots$ Fibonacci sequence.

$$f(n) + f(n+1) = \underline{f(n+2)}$$

$$f(0) = 1 = f(1)$$

Q: find $f(n)$, $n = 0, 1, 2, \ldots$

Soln: Apply z-transform to the 2^{nd} order difference equation, we get

$$z^2\left(F(z) - f(0) - f(1)\cdot\frac{1}{z}\right) = F(z) + z\left(F(z) - f(0)\right)$$

$$\Rightarrow \quad F(z)\left(z^2 - z + 1\right) = z^2 + \cancel{z} - \cancel{z}$$

$$z^2 - z + 1 = (z - a)(z - b)$$

$$z = \frac{1 \pm \sqrt{1-4}}{2}$$

$$z = \frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}.$$

$$a = \frac{(1 + i\sqrt{3})}{2}, \quad b = \frac{1 - i\sqrt{3}}{2}.$$

$$\Rightarrow \quad F(z) = \frac{z^2}{\underline{z^2 - z + 1}} = \frac{z}{(z-a)(z-b)}.$$

Inversion

$$f(n) = a^n * b^n.$$

$$= \sum_{m=0}^{n} a^{n-m} b^m$$

$$f(n) = \frac{a^{n+1} - b^{n+1}}{a-b}, \quad n = 0, 1, 2, \ldots,$$

$$f(n) = -i\,\frac{(1+i\sqrt{3})^{n+1} - (1-i\sqrt{3})^{n+1}}{2^{n+1}\sqrt{3}}, \quad n = 0, 1, 2, \ldots.$$

* Solve $f(n+2) - 3f(n+1) + 2f(n) = 0,$

$$f(0) = 1$$

$$f(1) = 2.$$

Soln: Z-transform gives

$$z^2\left(F(z) - 1 - 2\cdot\frac{1}{z}\right) - 3z\left(F(z) - 1\right) + 2F(n) = 0.$$

$$\Rightarrow \quad (z^2 - 3z + 2)\,F(z) = z^2 + 2z - 3z = z^2 - z = z(z-1)$$

$$\Rightarrow \quad F(z) = \frac{z\cancel{(z-1)}}{\cancel{(z-1)}\,(z-2)} = \frac{z}{z-2}, \qquad |z| > 2.$$

Inversion gives $\quad Z^{-1}(F(z)) = f(n) = Z^{-1}\left(\frac{z}{z-2}\right) = 2^n, \quad n = 0, 1, 2, \ldots$

* Solve $f(n+2) - f(n+1) + f(n) = 0, \quad n = 0, 1, 2, \ldots$

$f(0) = 1$

$f(1) = 2$.

Soln: Application of Z-transform gives

$$z^2\left(F(z) - 1 - 2\cdot\frac{1}{z}\right) - z\left(F(z) - 1\right) + F(z) = 0$$

$$F(z)\left(z^2 - z + 1\right) = z^2 + 2z - z = z(z+1).$$

$$\Rightarrow \quad F(z) = \frac{z(z+1)}{z^2 - z + 1}$$

Inversion of Z-transform gives

$$f(n) = Z^{-1}\left(\frac{z^2+z}{z^2-z+1}\right), \quad \underline{|z|>1}$$

$$= Z^{-1}\left(\frac{z^2-\frac{1}{2}z}{z^2-z+1} + \frac{\frac{1}{2}z+z}{z^2-z+1}\right), \quad |z|>1$$

$$= Z^{-1}\left(\frac{z^2-\frac{1}{2}z}{z^2-z+1}\right)+\sqrt{3}\, Z^{-1}\left(\frac{\sqrt{3}/2\, z}{z^2-z+1}\right), \quad |z|>1$$

$$f(n) = \underline{\cos\frac{n\pi}{3}+\sqrt{3}\sin\frac{n\pi}{3}}, \quad n=0,1,2,\ldots$$

$$Z(\cos nx)(z) = \underline{\frac{z(z-\cos x)}{z^2-2z\cos x+1}}, \quad |z|>1$$

$$2\cos x = 1$$

$$\cos x = 1 \Rightarrow x = \pi/3$$

$$\underline{Z\left(\cos\frac{n\pi}{3}\right)(z) = \frac{z^2-\frac{1}{2}z}{z^2-z+1}}$$

$$Z(\sin nx)(z) = \frac{z\sin x}{z^2-2z\cos x+1}$$

$$Z\left(\sin\frac{n\pi}{3}\right)(z) = \frac{z\cdot\frac{\sqrt{3}}{2}}{z^2-z+1}$$

* Solve $f(n+2) - 5f(n+1) + 6f(n) = 2^n$, $n = 0, 1, 2, \dots$

$f(0) = 1$

$f(1) = 0$.

Sol: Z-transform gives

$z^2(F(z) - 1) - 5z(F(z) - 1) + 6F(z) = \frac{z}{z-2}$, $|z| > 1$

$F(z)(z^2 - 5z + 6) = z^2 - 5z + \frac{z}{z-2}$

$F(z) = \frac{z(z-5)}{(z-2)(z-3)} + \frac{z}{(z-2)^2(z-3)}$, $|z| > 1$.

$F(z) = z\left[\frac{z-5}{(z-2)(z-3)} + \frac{1}{(z-2)^2(z-3)}\right]$, $|z| > 1$

$$F(z) = z\left[\frac{3}{z-2} - \frac{2}{z-3} + \frac{-1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{z-3}\right]$$

$$Z(n) = \frac{z}{(z-1)^2}$$

Inversion gives $f(n) = \underline{3 \cdot 2^n} - \underline{2\,3^n} - \underline{2^n} - Z^{-1}\left(\frac{z}{(z-2)^2}\right) + \underline{3^n}$

$$= 2^{n+1} - 3^n - \underline{Z^{-1}\left(\frac{z}{(z-2)^2}\right)}$$

$$= 2^{n+1} - 3^n - \frac{1}{2} \cdot 2^n.$$

$$\boxed{f(n) = 2^{n+1} - 3^n - 2^{n-1}, \quad n = 0, 1, 2, \ldots}$$

$$Z(n\,\underline{f(n)}) = -z\frac{d}{dz}(F(z))$$

$$Z(2^n) = \frac{z}{z-2} \checkmark$$

$$Z(\underline{n\;2^n}) = -z\frac{d}{dz}\left(\frac{z}{z-2}\right)$$

$$= -z\left(\frac{1}{z-2} + z\left(\frac{-1}{(z-2)^2}\right)\right)$$

$$= -z\left(\frac{\cancel{z}-2-\cancel{z}}{(z-2)^2}\right) = \frac{2z}{(z-2)^2}.$$

Property of Z-transform:

1. If $F(z) = Z(f(n))(z)$, then

(i) $Z\left(\dfrac{f(n)}{n}\right)(z) = \displaystyle\int_z^{\infty} \dfrac{F(z)}{z}\, dz$. ✓

$$\sum_{n=0}^{\infty} \frac{f(n)}{n} z^{-n} < \infty,$$

$f(0) = 0$

(ii) $Z\left(\dfrac{f(n)}{n+m}\right)(z) = z^{m}\displaystyle\int_z^{\infty} \dfrac{F(z)}{z^{m+1}}\, dz$; $m = 0, 1, 2, \ldots$

Proof:

(i) $$\int_z^{\infty} \frac{F(z)}{z}\, dz = \int_z^{\infty} \frac{\sum_{n=0}^{\infty} f(n) z^{-n}}{z}\, dz$$

$$= \int_z^{\infty} \sum_{n=0}^{\infty} f(n) z^{-n-1}\, dz$$

$$= \sum_{n=0}^{\infty} f(n) \int_{t}^{\infty} z^{-n-1} \, dt \checkmark$$

$$= \sum_{n=0}^{\infty} f(n) \cdot \frac{z^{-n}}{-n} \Bigg|_{z=t}^{\infty}$$

$$= \sum_{n=0}^{\infty} \frac{f(n)}{n} z^{-n} .$$

$$= Z\left(\frac{f(n)}{n}\right)(t)$$

(ii) $$z^{m} \int_{t}^{\infty} \frac{F(t)}{t^{m+1}} \, dt = z^{m} \int_{t}^{\infty} \sum_{n=0}^{\infty} f(n) \, z^{-n-m-1} \, dt$$

$$= z^{m} \sum_{n=0}^{\infty} f(n) \int_{z}^{\infty} z^{-n-m-1} \, dt$$

$$= \cancel{z^{m}} \sum_{n=0}^{\infty} f(n) \cdot \frac{z^{-n-\cancel{m}}}{-(n+m)} \Bigg|_{z}^{\infty}$$

$$= \sum_{n=0}^{\infty} \frac{f(n)}{n+m} z^{-n}, \qquad \forall \, m = 0, 1, 2, \ldots$$

$$= Z\left(\frac{f(n)}{n+m}\right)(z).$$

2. If $F(z) = Z(f(n))(z)$, then

(i) $Z\left(\sum_{k=0}^{n} f(k)\right)(z) = \frac{z}{z-1} F(z)$. ✓

(ii) $\sum_{k=0}^{\infty} f(k) = \lim_{z \to 1} z F(z) = F(1)$ ✓

proof: (i) Let $g(n) = \sum_{k=0}^{n} f(k)$.

$$\Rightarrow \quad g(n+1) - g(n) = f(n+1), \quad n = 0, 1, 2, \ldots$$

$$\Rightarrow \quad g(n) - g(n-1) = f(n); \quad n = 0, 1, 2, \ldots \qquad \left(\underline{g(-1) = 0} \right)$$

Application of Z-transform to the above difference equation gives

$$G(z) - z^{-1} G(z) = F(z).$$

$$G(z) \left(\frac{z-1}{z} \right) = F(z).$$

$$\Rightarrow \quad G(z) = \frac{z}{z-1} F(z) \checkmark$$

Inversion gives $\quad g(n) = Z^{-1}\left(\frac{z}{z-1} F(z)\right). \checkmark$

Final value Theorem :

$$\lim_{n\to\infty} f(n) = \lim_{z\to 1}\left((z-1) F(z)\right)$$

(ii) $$\sum_{k=0}^{\infty} f(k) = \lim_{n\to\infty} \sum_{k=0}^{n} f(k) = \lim_{z\to 1} (z-1)\, G(z)$$

$$= \lim_{z\to 1} \cancel{(z-1)} \cdot \frac{z}{\cancel{(z-1)}} F(z) \checkmark$$

$$\boxed{\sum_{k=0}^{\infty} f(k) = F(1).}$$

Summation of infinite Series.

1. S. that, by z-transform, $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

Since $Z(x^n f(n))(z) = Z(f(n))\left(\frac{z}{x}\right) = F\left(\frac{z}{x}\right)$, ✓

If $f(n) = \frac{1}{n!}$, $Z(f(n))(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = e^{1/z}$.

$$Z\left(\frac{x^n}{n!}\right) = e^{x/z}.$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^{x/z}\Big|_{z=1} = e^x.$$ ✓

2. S. that $$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \log(1+x).$$

Soln:

$$Z(x^{n+1})(z) = x\, Z(x^n)(z) = x \cdot \frac{z}{z-x} = \frac{zx}{z-x}.$$

$$Z\left(\frac{x^{n+1}}{n+1}\right)(z) = z \int_z^{\infty} \frac{\cancel{z}x}{z-x} \cdot \frac{1}{z^{\cancel{2}}}\, dz$$

$$= xz \int_z^{\infty} \frac{1}{z(z-x)}\, dz$$

$$= z \int_z^{\infty} \left(\frac{1}{z-x} - \frac{1}{z}\right) dz$$

$$= z \left(\log \frac{(z-x)}{z} \Big|_z^{\infty} \right)$$

$$= z\left(-\log\left(\frac{z-x}{z}\right)\right)$$

$$Z\left(\frac{x^{n+1}}{n+1}\right)(z) = -z\log\left(\frac{z-x}{z}\right).$$

Replace x by $-x$, we get

$$Z\left(\frac{(-x)^{n+1}}{n+1}\right)(z) = -z\log\left(\frac{z+x}{z}\right)$$

$$\Rightarrow \quad Z\left((-1)^n\frac{x^{n+1}}{n+1}\right)(z) = z\log\left(\frac{z+x}{z}\right)$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{x^{k+1}}{k+1} = \lim_{z\to 1} z^{2}\log\left(\frac{z+x}{z}\right) \doteq \log(1+x).$$

3. Find the sum $\underline{\sum_{n=0}^{\infty} a^n \sin nx}$.

Soln:

$$Z\left(a^n \sin nx\right)(z) = Z\left(\sin nx\right)\left(\frac{z}{a}\right)$$

$$= \frac{\frac{z}{a}\sin x}{\frac{z^2}{a^2} - 2\frac{z}{a}\cos x + 1}.$$

$$= \frac{z a \sin x}{z^2 - 2za\cos x + a^2}.$$

$$\sum_{n=0}^{\infty} a^n \sin nx = \lim_{z\to 1} z \cdot \frac{z a \sin x}{z^2 - 2za\cos x + a^2}.$$

$$= \frac{a \sin x}{1 - 2a\cos x + a^2} \checkmark$$