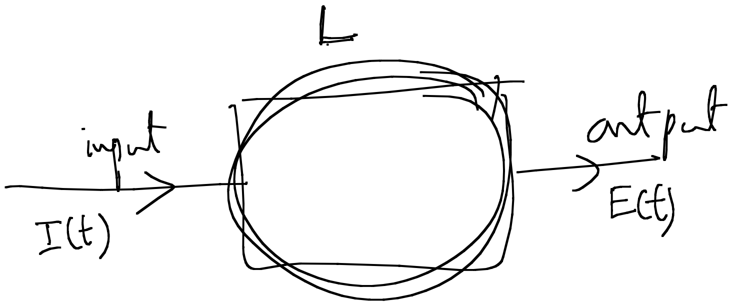
Transform Techniques for Engineers

Note Title 15-02-2018



Signal: $f(t)$ — a real or complex valued function of time.

$z = x + iy, \quad x, y \in \mathbb{R}.$

$z = r e^{i\theta}$ Euler representation.

eg: $\sin \omega t, \ \cos \omega t$ fundamental signals

Linear System: L is linear if $L(cf_1+f_2) = cLf_1 + Lf_2$

Repetitive phenomenon in natural and engg. Sciences

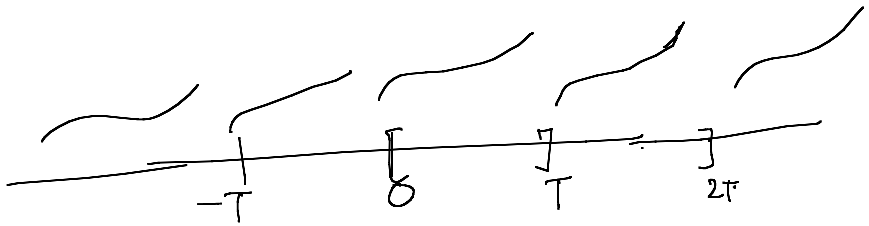
$[0, T]$ $[T, 2T]$ - - and so on.

periodic functions: A function $f(t)$ is periodic if

$f(t+T) = f(t)$, $\forall\, t$.

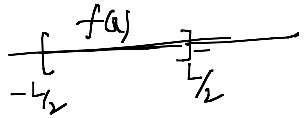
eg: $\sin t$, $\cos t$

$f(t)$



Fourier coefficients: Let $f(x)$ be a periodic function with period L. Then the frequency of $f(x)$ is $\omega_0 = \frac{2\pi}{L}$ (Fundamental).

The Fourier coefficients are defined as



for each $n = 0, 1, 2, 3, \ldots,$

Fourier transform of $f(x)$

$$\begin{cases} a_n = \dfrac{2}{L} \displaystyle\int_{-L/2}^{L/2} f(x) \cos(n \omega_0 x)\, dx \ \checkmark \\[2ex] b_n = \dfrac{2}{L} \displaystyle\int_{-L/2}^{L/2} f(x) \sin(n \omega_0 x)\, dx \ \checkmark \end{cases}$$

when They exist.

1–1 Correspondence

Fourier coefficients

periodic function $\longleftrightarrow$ $\{a_n, b_n\}_{n=0,1,2,\dots}$

$f(x)$

$(\longrightarrow)$ Fourier transform ✓

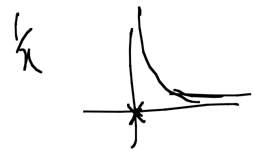
$(\longleftarrow)$ Inverse fourier transform or

Fourier Series

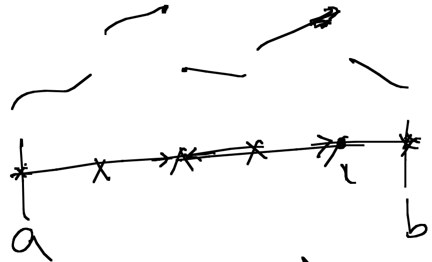
$$f(x) \overset{?}{=} \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 x) + b_n \sin(n\omega_0 x)\right] \quad (*)$$ ✓

Under some sufficient conditions on $f(x)$, $(*)$ is true for each x.

piecewise continuous function:

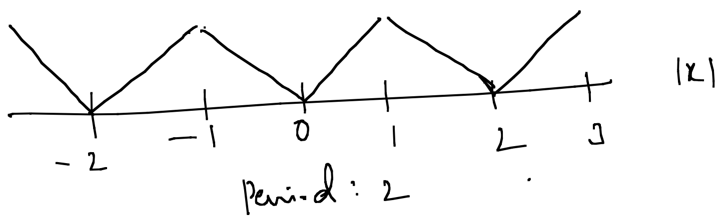


$$f'(x) = \lim_{h \to 0^{\pm}} \frac{f(x+h) - f(x)}{h}$$



A function with finite # of discontinuities (jump) is piecewise continuous function.

piecewise differentiable function: if $f'(x)$ is piecewise continuous function. then $f(x)$ is piecewise differentiable function:



Example: $f(x) = |x|, \quad [-1, 1]$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\omega_0 x + b_n \sin n\omega_0 x\right)$$

$$\omega_0 = \frac{2\pi}{L} = \pi$$

$$a_n = \frac{2}{2}\int_{-1}^{1} f(x) \cos n\pi x \, dx ; \quad n = 0, 1, 2, \ldots \quad \text{and} \quad b_n = \frac{2}{2}\int_{-1}^{1} f(x) \sin n\pi x \, dx , \quad n = 1, 2, 3, \ldots$$

$n = 0,$
$$a_0 = \int_{-1}^{1} |x| \, dx = \int_{0}^{1} x \, dx - \int_{-1}^{0} x \, dx = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$$

$n = 1, 2, 3, \ldots,$
$$a_n = \int_{-1}^{1} |x| \cos n\pi x \, dx = \int_{0}^{1} x \cos n\pi x \, dx - \int_{-1}^{0} x \cos n\pi x \, dx$$

$$= \cancel{\frac{\sin n\pi x}{n\pi} x \Big|_{0}^{1}} - \frac{1}{n\pi}\int_{0}^{1} \sin n\pi x \, dx - \cancel{\frac{\sin n\pi x}{n\pi} x \Big|_{-1}^{0}} + \frac{1}{n\pi}\int_{-1}^{0} \sin n\pi x \, dx$$

$$= \frac{1}{n\pi} \frac{\cos n\pi x}{n\pi}\Big|_{0}^{1} - \frac{1}{n\pi} \frac{\cos n\pi x}{n\pi}\Big|_{-1}^{0}$$

$$a_n = \frac{1}{n^2\pi^2}\left((-1)^n - 1\right) + \frac{1}{n^2\pi^2}\left(-1 + (-1)^n\right) = \frac{2}{n^2\pi^2}\left((-1)^n - 1\right), \quad n = 1, 2, 3, \ldots$$

$$b_n = \int_{-1}^{1} |x| \sin n\pi x \, dx = \int_0^1 x \sin n\pi x \, dx - \int_{-1}^{0} x \sin n\pi x \, dx$$

$$= \cancel{-\frac{\cos n\pi x}{n\pi} x \Big|_{x=0}^{1}} + \int_0^1 \frac{\cos n\pi x}{n\pi} \, dx + \cancel{\frac{\cos n\pi x}{n\pi} x \Big|_{-1}^{0}} - \int_{-1}^{0} \frac{\cos n\pi x}{n\pi} \, dx .$$

$$= \cancel{\frac{1}{n^2\pi^2} \sin n\pi x \Big|_0^1} - \cancel{\frac{1}{n^2\pi^2} \sin n\pi x \Big|_{-1}^{0}}$$

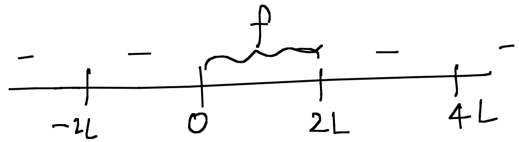
$$= 0 \, , \quad n = 1, 2, 3, \ldots$$

$|x|$ is even function, $b_n = 0$.

$$\Rightarrow \quad \underline{|x| = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \left((-1)^n - 1 \right) \cos n\pi x \, ;} \qquad x \in [-1, 1] .$$

Transform Techniques for Engineers

Let $f(x) = x(2L - x), \quad 0 \le x \le 2L.$



Fourier coefficients

$$a_n = \frac{2}{2L} \int_0^{2L} f(x) \cos n\omega_0 x \, dx, \quad \omega_0 = \frac{2\pi}{2L} = \frac{\pi}{L}$$

$$a_n = \frac{1}{L} \int_0^{2L} f(x) \cos \frac{n\pi x}{L} \, dx; \quad n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L} \int_0^{2L} f(x) \sin \frac{n\pi x}{L} \, dx, \quad n = 1, 2, 3, \ldots$$

$$a_0 = \frac{1}{L} \int_0^{2L} x(2L - x) \, dx = \frac{1}{L}\left(x^2 L - \frac{x^3}{3}\right)\Bigg|_0^{2L} = 4L^2 - \frac{8L^2}{3}$$

$$= 4L^2\left(1 - \frac{2}{3}\right) = \frac{4}{3}L^2.$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n=1,2,3,\ldots$$

$$= \frac{1}{\cancel{L}}\left[\cancel{\frac{f(x)\sin\frac{n\pi x}{L}}{\frac{n\pi}{\cancel{L}}}\Bigg|_0^{2L}} - \frac{\cancel{L}}{n\pi}\int_0^{2L}\sin\frac{n\pi x}{L}\,\underline{(2L-2x)}\,dx\right].$$

$$= \frac{(2L-2x)L}{n^2\pi^2}\cos\frac{n\pi x}{L}\Bigg|_0^{2L} + \frac{2L}{n^2\pi^2}\int_0^{2L}\cos\frac{n\pi x}{L}\,dx.$$

$$= \frac{-2L^2}{n^2\pi^2} - \frac{2L^2}{n^2\pi^2} + \cancel{\frac{2L}{n^2\pi^2}\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Bigg|_0^{2L}}$$

$$a_n = \frac{-4L^2}{n^2\pi^2}, \quad n=1,2,3,\ldots$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n=1,2,3,\ldots$$

$$= \frac{1}{\cancel{L}}\left[\cancel{-f(x)\frac{\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}} + \frac{\cancel{L}}{n\pi}\int_0^{2L} \cos\frac{n\pi x}{L}\cdot(2L-2x)\,dx \right]$$

$f(x) = 2Lx - x^2$

$$= \frac{1}{n\pi}\left[\cancel{(2L-2x)\frac{\sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}\Bigg|_0^{2L}} + \frac{2L}{n\pi}\int_0^{2L} \sin\frac{n\pi x}{L}\,dx \right]$$

$$= \frac{2L}{n^2\pi^2}\left[-\cos\frac{n\pi x}{L}\cdot\frac{L}{n\pi} \right]\Bigg|_0^{2L} = \frac{-2L^2}{n^3\pi^3}[1-1] = 0 \checkmark$$

$$(2L-x)x \overset{?}{=} \frac{2}{3}L^2 - \sum_{n=1}^{\infty} \frac{4L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right). \quad \text{(Fourier series)}$$

$$(2L-x)x = \frac{2}{3}L^2 - \frac{4L^2}{\pi^2}\left(\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi x}{L}\right)}{n^2} \right); \quad 0 \le x \le 2L.$$

put $x = 0$, $\quad 0 = \frac{2}{3}L^2 - \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\cancel{2}}{3} \cdot \frac{\pi^2}{\cancel{4}\,2} = \frac{\pi^2}{6} \checkmark$$

<u>put $x = L$</u>,
$$\cancel{L^2} = \frac{2}{3}\cancel{L^2} - \frac{4\cancel{L^2}}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

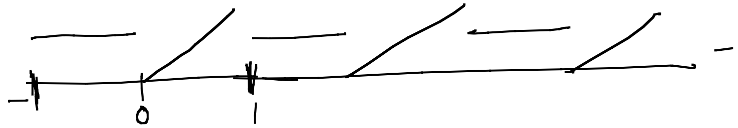
$$-\frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\Rightarrow -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{\pi^2}{3\cdot 4} = \frac{\pi^2}{12}. \checkmark$$

$$\Rightarrow \boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}} \checkmark$$

Example:

$$f(x) = \begin{cases} k, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$$



is a piecewise continuous function with period '2'.

$$\omega_0 = \frac{2\pi}{2} = \pi$$

$$\left.\begin{aligned} a_n &= \int_{-1}^{1} f(x) \cos(n\pi x)\, dx \\ b_n &= \int_{-1}^{1} f(x) \sin(n\pi x)\, dx \end{aligned}\right\} n = 0, 1, 2, 3, ---$$

$$a_n = k \int_{-1}^{0} \cos n\pi x \, dx + \int_{0}^{1} x \cos n\pi x \, dx; \quad n = 0, 1, 2, ---$$

$$a_0 = k + \frac{1}{2} \checkmark$$

$$a_n = k \cdot \cancel{\frac{\sin n\pi x}{n\pi}\Big|_{-1}^{0}} + \cancel{x\frac{\sin n\pi x}{n\pi}\Big|_{0}^{1}} - \frac{1}{n\pi}\int_0^1 \sin n\pi x \, dx, \quad n = 1, 2, 3, \ldots$$

$$a_n = +\frac{1}{n^2\pi^2}\cos n\pi x\Big|_0^1 = \frac{1}{n^2\pi^2}\left[(-1)^n - 1\right], \quad n = 1, 2, 3, \ldots \checkmark$$

$$b_n = k\int_{-1}^{0} \sin n\pi x \, dx + \int_0^1 x \sin n\pi x \, dx, \quad n = 1, 2, 3, \ldots$$

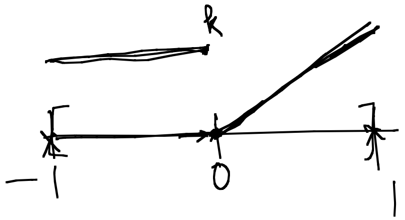
$$= k\left(-\frac{\cos n\pi x}{n\pi}\right)\Big|_{-1}^{0} + -\frac{x\cos n\pi x}{n\pi}\Big|_{-1}^{0} + \int_0^1 \frac{\cos n\pi x}{n\pi} \, dx.$$

$$= \frac{k(-1+(-1)^n)}{n\pi} + \frac{(-1)^n}{n\pi} + \cancel{\frac{\sin n\pi x}{n^2\pi^2}\Big|_0^1}$$

$$b_n = -\frac{k}{n\pi} + \frac{(-1)^n}{n\pi}(k+1) \checkmark$$

$$f(x) = \begin{cases} k, & -1 \le x < 0 \\ x, & 0 < x \le 1 \end{cases} = \frac{2k+1}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}\left((-1)^n - 1\right)\cos n\pi x + \left(\frac{(-1)^n(k+1)}{n\pi} - \frac{k}{n\pi}\right)\sin n\pi x. \checkmark$$

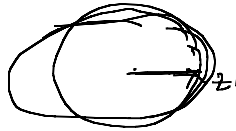
(Fourier Series)



$\underline{x=0}$

$$\left.\begin{aligned} f(0^-) &= k \\ f(0^+) &= 0 \end{aligned}\right\} \quad \frac{f(0^+) + f(0^-)}{2} = \frac{k}{2}.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 x + b_n \sin n\omega_0 x\right), \quad \omega_0 = \frac{2\pi}{L}$$



$z(t)$

$= x(t) + i\,y(t)$

$= r(t)\,e^{i\theta}$

$0 \le \theta < 2\pi$

where

$$\left.\begin{aligned} a_n &:= \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos n\omega_0 x\,dx, \quad n = 0, 1, 2, \dots \\ b_n &:= \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin n\omega_0 x\,dx; \quad n = 1, 2, 3, \dots \end{aligned}\right\}\ \text{Fourier Coefficients}$$

$$\Updownarrow$$

$$\boxed{\cos\theta + i\sin\theta = e^{i\theta}}$$

$$\cos n\omega_0 x = \frac{e^{in\omega_0 x} + e^{-in\omega_0 x}}{2}, \quad \sin n\omega_0 x = \frac{e^{in\omega_0 x} - e^{-in\omega_0 x}}{2i}$$

clearly $b_0 = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{in\omega_0 x} + e^{-in\omega_0 x}}{2} + b_n \frac{e^{in\omega_0 x} - e^{-in\omega_0 x}}{2i} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{in\omega_0 x}\left(\frac{a_n}{2} - i\frac{b_n}{2}\right) + e^{-in\omega_0 x}\left(\frac{a_n}{2} + i\frac{b_n}{2}\right)$$

$$= \frac{a_0 - ib_0}{2} + \sum_{n=1}^{\infty} \left(e^{in\omega_0 x} \frac{a_n - ib_n}{2} + e^{-in\omega_0 x} \frac{a_n + ib_n}{2} \right)$$

$$= \sum_{n=0}^{\infty} e^{in\omega_0 x} \frac{a_n - ib_n}{2} + \sum_{n=1}^{\infty} e^{-in\omega_0 x} \frac{a_n + ib_n}{2}.$$

Let $c_n := \frac{a_n - ib_n}{2}, \quad n = 0, 1, 2, 3, \ldots$

$c_{-n} := \frac{a_n + ib_n}{2}, \quad n = 1, 2, 3, \ldots$

$$f(x) = \sum_{n=0}^{\infty} c_n e^{in\omega_0 x} + \sum_{n=1}^{\infty} c_{-n} e^{-in\omega_0 x}.$$

$$= \sum_{n=0}^{\infty} c_n e^{in\omega_0 x} + \sum_{n=-1}^{-\infty} c_n e^{in\omega_0 x}$$

Complex Fourier Series

$$\boxed{f(x) \overset{?}{=} \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}}$$ ✓ where $c_n = \frac{a_n - ib_n}{2}$

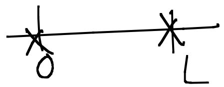
$$c_n = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}}{L}\int_{-L/2}^{L/2} f(x)\cos n\omega_0 x\,dx - i\,\frac{\cancel{2}}{L}\int_{-L/2}^{L/2} f(x)\sin n\omega_0 x\,dx\right]$$

$$= \frac{1}{L}\int_{-L/2}^{L/2} f(x)\left(\cos n\omega_0 x - i\cdot\sin n\omega_0 x\right)dx$$

✓ $c_n := \frac{1}{L}\int_{-L/2}^{L/2} f(x)\,\underline{e^{-in\omega_0 x}}\,dx$ ✓ (Fourier coefficient)

$\omega_0 = \frac{2\pi}{L}$ $\cos n\omega_0 x$, $\sin n\omega_0 x$

$$a_n, b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x) \begin{matrix}\cos\\ \sin\end{matrix}(n\omega_0 x)\, dx$$



Periodic S-L system

$$y'' + \lambda y = (1\cdot y')' + \lambda \cdot 1 \cdot y = 0, \quad -\frac{L}{2} < x < L/2$$

λ is a real parameter

$$\langle f, g\rangle := \int_{-L/2}^{L/2} f(x)\, g(x)\, dx$$

B.C's: $y(-L/2) = y(L/2)$

$y'(-L/2) = y'(L/2)$

non-zero Solutions: $\cos \frac{n 2\pi}{L} x$, $\sin \frac{n2\pi}{L} x$, $n = 0, 1, 2, \dots$

1, $\cos(n\omega_0 x)$, $\sin(n\omega_0 x)$, $n = 1, 2, 3, \dots$

$$\int_{-L/2}^{L/2} \cos n\omega_0 x \cdot \sin n\omega_0 x \, dx = 0; \ \forall n$$

$$\int_{-L/2}^{L/2} \binom{\cos}{\sin}^2 n\omega_0 x \, dx = \frac{L}{2}, \quad n = 0, 1, 2, \ldots$$

Complet set:
$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\omega_0 x + b_n \sin n\omega_0 x \right)$$

where $\langle a_0 \cdot 1 \rangle = \langle f(x), 1 \rangle$

$$\Rightarrow \int_{-L/2}^{L/2} a_0 \, dx = a_0 \frac{L}{2} = \int_{-L/2}^{L/2} f(x) \, dx$$

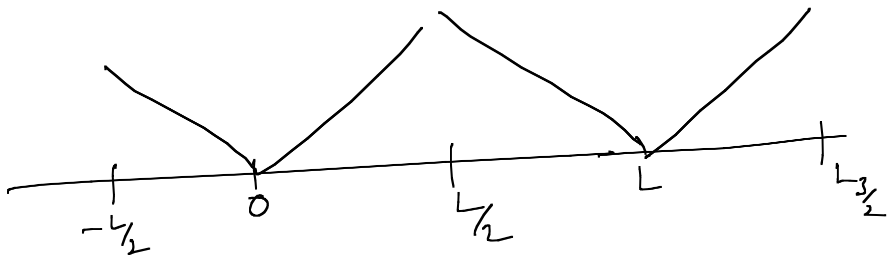
$$\Rightarrow a_0 = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \, dx.$$

$$\left\langle f(x), \begin{smallmatrix}\cos \\ \sin\end{smallmatrix} n\omega_0 x \right\rangle = \begin{smallmatrix}a_n \\ (b_n)\end{smallmatrix} \left\langle \begin{smallmatrix}\cos \\ \sin\end{smallmatrix} n\omega_0 x, \begin{smallmatrix}\cos \\ \sin\end{smallmatrix} n\omega_0 x \right\rangle$$

$$= \frac{L}{2} \begin{smallmatrix}a_n \\ (b_n)\end{smallmatrix}$$

$$\Rightarrow \checkmark a_n := \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos n\omega_0 x \, dx$$

$$b_n := \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin n\omega_0 x \, dx$$



For example, $f(x) = |x|$, $-\frac{L}{2} \le x < \frac{L}{2}$

period is L., $\omega_0 = \frac{2\pi}{L}$

Fourier coefficients $C_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-in\frac{2\pi}{L}x} dx$, $n = 0, \pm1, \pm2, \cdots$

$$= \frac{-1}{L}\int_{-L/2}^{0} x\, e^{-in\omega_0 x} dx + \frac{1}{L}\int_{0}^{L/2} x\, e^{-in\omega_0 x} dx$$

$$= \frac{1}{L}\int_0^{L/2} x\, e^{in\omega_0 x}\, dx + \frac{1}{L}\int_0^{L/2} x\, e^{-in\omega_0 x}\, dx. \qquad n\omega_0 = n.\frac{2\pi}{L}$$

$$= \frac{2}{L}\int_0^{L/2} x \underline{\cos n\omega_0 x}\, dx = \frac{2}{L} \cancel{\frac{\sin n\omega_0 x}{n\omega_0}\, x \Big|_0^{L/2}} - \frac{2}{L\, n\omega_0}\int_0^{L/2} \sin n\omega_0 x \cdot dx$$

$$= \frac{1}{n\pi}\, \frac{\cos n\omega_0 x}{n\omega_0}\Big|_0^{L/2}$$

$$= \frac{L}{n^2\pi^2 2}\left[\cos n\frac{2\pi}{\cancel{L}}\cdot\frac{\cancel{L}}{\cancel{2}} - 1\right]$$

$$= \frac{L}{2n^2\pi^2}\left((-1)^n - 1\right)$$

$$\Rightarrow |x| \stackrel{?}{=} \underline{\sum_{n=-\infty}^{\infty} \frac{L}{2n^2\pi^2}\left((-1)^n - 1\right) e^{in\frac{2\pi}{L}x}}, \quad \underline{-\tfrac{L}{2} \le x \le \tfrac{L}{2}}$$

Theorem:
Let $f(x)$ be piecewise continuous periodic function with period L.

Then. $$\boxed{\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{L} \int_{-L/2}^{L/2} |f(x)|^2 \, dx.}$$ ✓ (Bessel's inequality)

Proof: Fourier Series $\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}$, ✓ $c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-in\omega_0 x} dx$.

Let $$S_n(x) = \sum_{k=-n}^{n} c_k e^{ik\omega_0 x}.$$ ✓

Observe that $$\int_{-L/2}^{L/2} [f(x) - S_n(x)] e^{-in\omega_0 x} dx$$ ✓

$$= \int_{-L/2}^{L/2} f(x) e^{-in\omega_0 x} dx - \int_{-L/2}^{L/2} \sum_{k=-n}^{n} c_k e^{ik\omega_0 x} \cdot e^{-in\omega_0 x} dx$$

$$= L C_n - \sum_{k=-n}^{n} C_k \int_{-L/2}^{L/2} e^{i(k-n)\omega_0 x}\, dx \qquad (k-n) = l \neq 0 \quad \omega_0 = \frac{2\pi}{L}$$

$$\int_{-L/2}^{L/2} e^{il\omega_0 x}\, dx = \left. \frac{e^{il\omega_0 x}}{i\omega_0} \right|_{-L/2}^{L/2} = e^{il\frac{2\pi}{L}\frac{L}{2}} - e^{-il\pi} = (-1)^l - (-1)^l = 0 \checkmark$$

$$= L C_n - C_n L = 0 \checkmark$$

Note that $\displaystyle\int_{-L/2}^{L/2} [f(x) - S_n(x)]\, \overline{S_n(x)}\, dx$

$$= \sum_{k=-n}^{n} \overline{C_k} \int_{-L/2}^{L/2} (f(x) - S_n(x))\, e^{-ik\omega_0 x}\, dx$$

$$= 0 \checkmark \implies \int_{-L/2}^{L/2} f(x)\, \overline{S_n(x)}\, dx = \int_{-L/2}^{L/2} |S_n(x)|^2\, dx.$$

$$0 \leq \int_{-L/2}^{L/2} \left(f(x) - S_n(x)\right) \overline{\left(f(x) - S_n(x)\right)} \, dx = \int_{-L/2}^{L/2} \left(f(x) - S_n(x)\right) \overline{f(x)} \, dx$$

$$= \int_{-L/2}^{L/2} |f(x)|^2 \, dx - \int_{-L/2}^{L/2} S_n(x) \overline{f(x)} \, dx$$

$$= \int_{-L/2}^{L/2} |f(x)|^2 \, dx - \underline{\underline{\int_{-L/2}^{L/2} |S_n(x)|^2 \, dx}} .$$

$$\underline{\underline{k = m}}$$

$$\int_{-L/2}^{L/2} S_n(x) \overline{S_n(x)} \, dx = \int_{-L/2}^{L/2} \underline{\sum_{k=-n}^{n} c_k e^{ik\omega_0 x}} \cdot \underline{\sum_{m=-n}^{n} \overline{c}_m e^{-im\omega_0 x}} \, dx .$$

$$= \sum_{k=-n}^{n} |c_k|^2 \int_{-L/2}^{L/2} dx = \underline{\underline{L \cdot \sum_{k=-n}^{n} |c_k|^2}} .$$

$$0 \le \int_{-L/2}^{L/2} |f(x)|^2 dx - L \sum_{k=-n}^{n} |C_k|^2, \quad \forall n.$$

$$\Rightarrow \sum_{k=-n}^{n} |C_k|^2 \le \frac{1}{L} \int_{-L/2}^{L/2} |f(x)|^2 dx, \quad \forall n.$$

$$\Rightarrow \boxed{\sum_{k=-\infty}^{\infty} |C_k|^2 \le \frac{1}{L} \int_{-L/2}^{L/2} |f(x)|^2 dx.} \checkmark$$

If $\sum_{n=0}^{\infty} a_n < \infty$, then $\lim a_n = 0$ ✓

Proof: Suppose $\lim a_n = L$. $L > 0$

$|a_n - L| < \frac{L}{2}$, $n \ge N$.

$\Rightarrow 0 < L - \frac{L}{2} < a_n < L + \frac{L}{2}$, $\forall n \ge N$

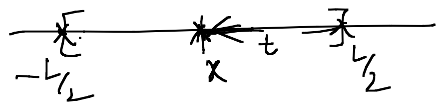
$\Rightarrow \sum_{n=0}^{\infty} a_n = \infty$ ✓

Corollary: If $\int_{-L/2}^{L/2} |f(x)|^2 dx < \infty$, then $\sum_{n=-\infty}^{\infty} |C_n|^2 < \infty$ ✓

$$\Rightarrow \lim_{n \to \infty} C_n = 0 \checkmark$$ (Riemann-Lebesgue Lemma).

Theorem: If $f(x)$ is a piecewise differentiable periodic function with period L;

then
$$\sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 x} = \frac{1}{2}\left(f(x^+)+f(x^-)\right); \quad x \in \left(-\frac{L}{2}, \frac{L}{2}\right).$$
$$= f(x), \quad x \in \left(-\frac{L}{2}, \frac{L}{2}\right).$$



$$f(x) = f(x^+) = \lim_{t\to x^+} f(t)$$
$$f(x) = f(x^-) = \lim_{t\to x^-} f(t)$$

Proof: Let
$$S_n(x) = \sum_{k=-n}^{n} C_k e^{ik\omega_0 x} \checkmark$$
$$= \sum_{k=-n}^{n} \frac{1}{L}\int_{-L/2}^{L/2} f(t) e^{-ik\omega_0 t} dt \, e^{ik\omega_0 x}$$
$$= \frac{1}{L}\int_{-L/2}^{L/2} f(t) \cdot \sum_{k=-n}^{n} e^{-ik\omega_0 (t-x)} dt.$$
$$= \frac{1}{L}\int_{0}^{L/2} f(t) \sum_{k=-n}^{n} e^{-ik\omega_0 (t-x)} dt + \frac{1}{L}\int_{-L/2}^{0} f(t) \sum_{k=-n}^{n} e^{-ik\omega_0 (t-x)}.$$

Let $D_n(x) = \sum_{k=-n}^{n} e^{-ik\omega_0 x} = e^{in\omega_0 x} + e^{i(n-1)\omega_0 x} + \cdots + e^{-in\omega_0 x}$ ✓

$$= e^{in\omega_0 x}\left(1 + e^{-i\omega_0 x} + e^{-i2\omega_0 x} + \cdots + e^{-i2n\omega_0 x}\right)$$

$$= e^{in\omega_0 x} \cdot \left(\frac{1 - e^{-i\omega_0 x(2n+1)}}{1 - e^{-i\omega_0 x}}\right)$$

$$= \frac{e^{in\omega_0 x} - e^{-i(n+1)\omega_0 x}}{1 - e^{-i\omega_0 x}}$$

$$= \frac{e^{i\frac{\omega_0}{2}x}\left(e^{in\omega_0 x} - e^{-i(n+1)\omega_0 x}\right)}{\left(e^{i\frac{\omega_0}{2}x} - e^{-i\frac{\omega_0}{2}x}\right)}$$

$$= \frac{e^{i(n+\frac{1}{2})\omega_0 x} - e^{-i(n+\frac{1}{2})\omega_0 x}}{e^{i\frac{\omega_0}{2}x} - e^{-i\frac{\omega_0}{2}x}}.$$

$$D_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)\omega_0 x}{\sin \frac{\omega_0}{2} x} \checkmark$$

$D_n(-x) = D_n(x)$. even function $\checkmark$

$$S_n(x) = \frac{1}{L}\int_{-L/2}^{L/2} f(t)\, D_n(t-x)\, dt$$

$t - x = x'$

$dt = dx'$

$$= \frac{1}{L}\int_{-L/2}^{L/2} f(x+x')\, D_n(x')\, dx'$$

$$= \frac{1}{L}\int_{-L/2}^{0} f(x+x')\, D_n(x')\, dx' + \frac{1}{L}\int_{0}^{L/2} f(x+x')\, D_n(x')\, dx'$$

$x' = -t$

$$S_n(x) = \frac{1}{L}\int_0^{L/2} f(x-t)\, D_n(t)\, dt + \frac{1}{L}\int_0^{L/2} f(x+t)\, D_n(t)\, dt$$

$$= \frac{1}{L}\int_0^{L/2} \left[f(x+t) + f(x-t)\right] \frac{\sin\left(n+\frac{1}{2}\right)\omega_0 t}{\sin\frac{\omega_0}{2}t}\, dt$$

$$\lim_{t\to 0} \frac{f(x+t) - f(x^+)}{t} = f'(x^+) < \infty$$

$$\lim_{t\to 0} \frac{f(x-t) - f(x^-)}{t} = f'(x^-) < \infty$$

$$= \frac{1}{L}\int_0^{L/2} \left[f(x+t) - f(x^+) + f(x-t) - f(x^-)\right] \frac{\sin\left(n+\frac{1}{2}\right)\omega_0 t}{\sin\frac{\omega_0}{2}t}\, dt + \frac{1}{L}\int_0^{L/2} \left(f(x^+) + f(x^-)\right) \cdot \frac{\sin\left(n+\frac{1}{2}\right)\omega_0 t}{\sin\frac{\omega_0}{2}t}\, dt$$

$$S_n(x) = \frac{1}{L}\int_0^{L/2} \left(\frac{\left[f(x+t) - f(x^+) + f(x-t) - f(x^-)\right]}{t} \cdot \frac{t}{\sin\frac{\omega_0}{2}t}\right) \cdot \sin\left(n+\frac{1}{2}\right)\omega_0 t\, dt \checkmark$$

$$+ \frac{1}{L}\left(f(x^+) + f(x^-)\right)\int_0^{L/2} \frac{\sin\left(n+\frac{1}{2}\right)\omega_0 t}{\sin\frac{\omega_0 t}{2}}\, dt$$

Observe that $\displaystyle\int_0^{L/2} \frac{\sin (n+\frac{1}{2})\omega_0 t}{\sin \frac{\omega_0 t}{2}}\, dt = \frac{1}{2}\int_{-L/2}^{L/2} \frac{\sin (n+\frac{1}{2})\omega_0 t}{\sin \frac{\omega_0 t}{2}}\, dt$

$$= \frac{1}{2}\int_{-L/2}^{L/2} \sum_{k=-n}^{n} e^{-ik\omega_0 t}\, dt$$

$$= \frac{1}{2}\sum_{k=-n}^{n} \int_{-L/2}^{L/2} e^{-ik\omega_0 t}\, dt$$

$$= \frac{1}{2}\int_{-L/2}^{L/2} dt = \frac{L}{2}.$$

Let $\displaystyle Q(t) = \frac{f(x+t) - f(x^+) + f(x-t) - f(x^-)}{t} \cdot \frac{t}{\sin\frac{\omega_0 t}{2}}$

$\forall\ t \in (0, \frac{L}{2})$.

$$Q(t) \quad f(\tfrac{L}{2}+t) = \underbrace{\frac{f(-\frac{L}{2}+t) - f(-\frac{L}{2}) + f(\frac{L}{2}-t) - f(\frac{L}{2})}{2}}$$

$$+ \frac{f(-\frac{L}{2}) + f(\frac{L}{2})}{2}.$$

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{1}{L}\int_0^{L/2} Q(t)\sin\left(n+\frac{1}{2}\right)\omega_0 t\,dt + \underbrace{\frac{1}{2}\left(f(x^+)+f(x^-)\right)}.$$

$$\lim_{t\to 0} \left(f'(x^+) - f'(x^-)\right)\frac{t}{\sin\frac{\omega_0 t}{2}} \to \frac{L}{\pi}\left(f'(x^+)-f'(x^-)\right)$$

$$\lim_{t\to 0} Q(t) < \infty, \qquad \int_{-L/2}^{L/2} |Q(t)|^2 < \infty,$$

$$\therefore \lim_{n\to\infty} \frac{1}{L}\int_0^{L/2} Q(t)\sin\left(n+\frac{1}{2}\right)\omega_0 t\,dt$$

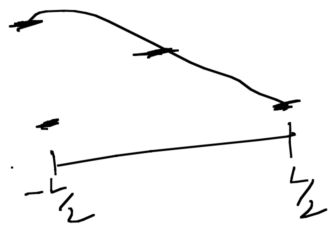
$$= \lim_{n\to\infty} \frac{1}{2L}\int_{-L/2}^{L/2} Q(t)\sin\left(n+\frac{1}{2}\right)\omega_0 t\,dt$$

$$= \lim_{n\to\infty} \frac{1}{4}\left[\underbrace{\frac{2}{L}\int_{-L/2}^{L/2} \left(\underbrace{Q(t)\cos\frac{\omega_0 t}{2}}_{g}\right)\sin n\omega_0 t\,dt} + \underbrace{\frac{2}{L}\int_{-L/2}^{L/2} \left(\underbrace{Q(t)\sin\frac{\omega_0 t}{2}}_{g}\right)\cos n\omega_0 t\,dt}\right]$$

$$= \frac{1}{4}\times 0 + \frac{1}{4}\times 0$$

$$\boxed{\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = \frac{1}{2}\left(f(x^+)+f(x^-)\right) = f(x),\quad x\in\left(-\frac{L}{2},\frac{L}{2}\right)}$$

If $x = \pm \frac{L}{2}$, $\lim_{n\to\infty} S_n(\frac{L}{2}) = \frac{f(-\frac{L}{2}) + f(\frac{L}{2})}{2}$ ✓



$$\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = \begin{cases} f(x), & x \in (-\frac{L}{2}, \frac{L}{2}) \\ \frac{f(\frac{L}{2}) + f(-\frac{L}{2})}{2}, & \text{if } x = \pm\frac{L}{2} \end{cases}$$

If $f(-\frac{L}{2}) = f(\frac{L}{2})$, then

$$\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = f(x),$$

$\forall x \in [-\frac{L}{2}, \frac{L}{2}]$

Cor:

Uniqueness of Fourier Series: Let $f(x)$ and $g(x)$ be two piecewise differentiable periodic functions in $[-\frac{L}{2}, \frac{L}{2}]$ with Fourier coefficients f_n, g_n such that $f_n = g_n$, $\forall n$.

Then $f(x) = g(x)$, for all x at which f and g are continuous.

Proof: If $f_n = g_n \;\forall\, n$, $\quad f_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-in\omega_0 x}\, dx$ ✓ (g_n, g)

then $C_n = f_n - g_n = 0$ ✓ $\quad C_n = f_n - g_n = \frac{1}{L}\int_{-L/2}^{L/2} (f-g)(x)\, e^{-in\omega_0 x}\, dx$

$$0 = \sum_{n=-\infty}^{\infty} C_n\, e^{in\omega_0 x} = f(x) - g(x)$$

$$\Rightarrow f(x) = g(x), \quad \forall\, x \in \left(-\frac{L}{2}, \frac{L}{2}\right).$$

Cor: If $f'(x)$ is piecewise continuous then

$$\lim_{n\to\infty} n \cdot f_n = 0. \quad \text{ie, } f_n \sim O\left(\frac{1}{n^2}\right).$$

Proof:

$$f'_n = \frac{1}{L}\int_{-L/2}^{L/2} f'(x)\, e^{in\omega_0 x}\, dx = \cancel{\frac{1}{L} f(x)\, e^{in\omega_0 x}\Big|_{-L/2}^{L/2}} - \underbrace{\frac{in\omega_0}{L}\int_{-L/2}^{L/2} f(x)\, e^{in\omega_0 x}\, dx}$$

$$= -\frac{in\, 2\pi}{L^2} \cdot f_n.$$

$$0 = \lim_{n\to\infty} f_n' = \lim_{n\to\infty} n f_n \left(-\frac{in2}{l^2}\right) = \left(-\frac{in2}{l^2}\right) \lim_{n\to\infty} n f_n$$

$$\Rightarrow \lim_{n\to\infty} n f_n = 0.$$

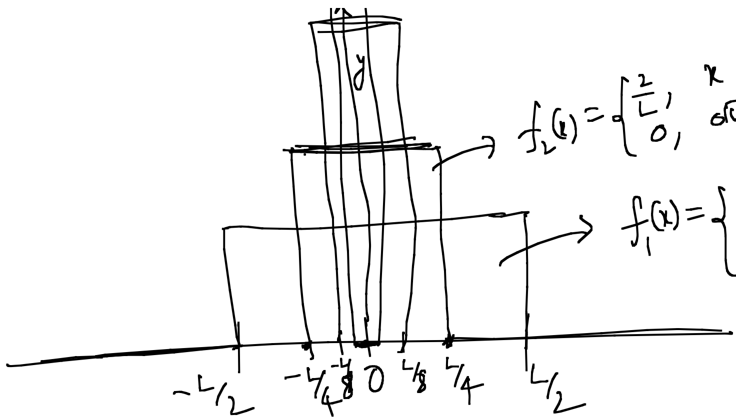
$\cancel{\delta : C_c^\infty \to \mathbb{R}}$ distribution.

$\cancel{\delta(f) = f(0)}$ ✓

δ - function (a generalized function)

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0. \end{cases}$$ ✓

$$\delta(x) = \lim_{n\to\infty} f_n(x),$$ for some functions $f_n(x)$.



$$f_2(x) = \begin{cases} \frac{2}{L}, & x \in [-\frac{L}{4}, \frac{L}{4}] \\ 0, & \text{otherwise} \end{cases}$$ ✓

$$f_1(x) = \begin{cases} \frac{1}{L}, & x \in [-\frac{L}{2}, \frac{L}{2}] \\ 0, & \text{otherwise} \end{cases}$$ ✓

$$\int_{-\infty}^{\infty} f_1(x)\, dx = \underline{1}$$ ✓

$$\int_{-\infty}^{\infty} f_2(x)\, dx = \int_{-L/4}^{L/4} \frac{2}{L}\, dx = 1$$ ✓

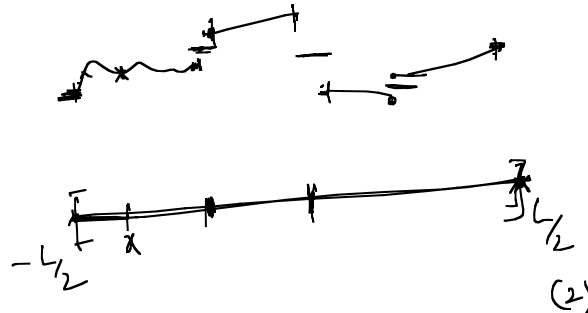
$$f_n(x) = \begin{cases} \frac{2^{n-1}}{L}, & x \in [-\frac{L}{2^n}, \frac{L}{2^n}] \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, 2, 3, \ldots$$ ✓

$$\int_{-\infty}^{\infty} f_n(x)\, dx = 1.$$

$$\underline{\int_{-\infty}^{\infty} \delta(x)\, dx} = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx$$

$$= \lim_{n\to\infty} 1 = 1$$ ✓

$$\lim_{n\to\infty} f_n(x) = \begin{cases} \infty, & x = 0 \\ 0, & \text{otherwise} \end{cases} =: \delta(x)$$



f has only finitely many jump discontinuities.

(2)

$$\left| c_n e^{i\omega_0 n x} \right| \le |c_n|$$

$\Rightarrow \sum_{n=-\infty}^{\infty} c_n e^{i\omega_0 n x}$ cges uniformly by M-Test.

Theorem:

If $f(x)$ is a piecewise continuous periodic function with period L. ie, $x \in [-L/2, L/2]$. and assume that $\sum^{\infty} c_n e^{i\omega_0 n x}$ cges pointwise.

Then
$$\sum_{n=-\infty}^{\infty} c_n e^{i\omega_0 n x} = \begin{cases} f(x), & \text{if } f \text{ is continuous at } x \\ \dfrac{f(x^+) + f(x^-)}{2}, & \text{if } f \text{ is not continuous at } x. \end{cases}$$

where $c_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x)\, e^{-i\omega_0 n x}\, dx$.

Defn: (δ- function)

If $\delta(x-t) = \begin{cases} \infty, & x = t \\ 0, & x \neq t \end{cases}$ and $\int_{-\infty}^{\infty} \delta(x-t)\, dt = 1$,

then $\delta(x-t)$ is called a δ- function.

$$\int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x-t)\, f(t)\, dt$$

$$= \lim_{n\to\infty} \int_{-L/2n}^{L/2n} \frac{2}{L}\, n^{-1}\, f(t)\, dt = f(x).$$

Result: $\int_{-\infty}^{\infty} \delta(x-t)\, f(t)\, dt = f(x)$

Proof: Let $S(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}$.

$$= \sum_{n=-\infty}^{\infty} \frac{1}{L} \int_{-L/2}^{L/2} f(t)\, e^{-in\omega_0 t}\, dt\; e^{in\omega_0 x}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{L} \int_{-L/2}^{L/2} f(t)\, e^{in\omega_0 (x-t)}\, dt.$$

$$= \frac{1}{L}\int_{-L/2}^{L/2} f(t)\,dt + \lim_{k\to\infty}\sum_{\substack{n=-k \\ n\neq 0}}^{k} \frac{1}{L}\int_{-L/2}^{L/2} f(t)\, e^{in\omega_0(x-t)}\, dt \checkmark$$

$$= \frac{1}{L}\int_{-L/2}^{L/2} f(t)\,dt + \lim_{k\to\infty}\frac{2}{L}\sum_{n=1}^{k} \int_{-L/2}^{L/2} f(t)\cos n\omega_0(x-t)\, dt \checkmark$$

$$= \frac{1}{L}\int_{-L/2}^{L/2} f(t)\,dt + \frac{2}{L}\lim_{k\to\infty}\sum_{n=1}^{k} \int_{-L/2}^{L/2} f(t)\cos n\omega_0(x-t)\, dt$$

$$= \int_{-L/2}^{L/2} f(t)\left[\frac{1}{L} + \frac{2}{L}\lim_{k\to\infty}\sum_{n=1}^{k}\cos n\omega_0(x-t)\right] dt$$

$$S(x) = \int_{-L/2}^{L/2} f(t)\underbrace{\left[\frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\cos n\omega_0(x-t)\right]}_{\delta(x-t).} dt \checkmark, \quad x \in \left[-\frac{L}{2}, \frac{L}{2}\right]$$

$$\frac{2}{L}\lim_{k\to\infty}\sum_{n=1}^{k}\int_{-L/2}^{L/2} f(t)\cos n\omega_0(x-t)\, dt$$

$$= \frac{2}{L}\lim_{k\to\infty}\int_{-L/2}^{L/2} f(t)\left(\sum_{n=1}^{k}\cos n\omega_0(x-t)\right) dt$$

$$= \frac{2}{L}\int_{-L/2}^{L/2} f(t)\left(\sum_{n=1}^{\infty}\cos n\omega_0(x-t)\right) dt$$

Then $$S(x) = \int_{-L/2}^{L/2} f(t)\, \delta(x-t)\, dt = \boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}}$$

Let $$D(x-t) = \frac{1}{L} + \frac{2}{L} \sum_{n=1}^{\infty} \cos n\omega_0 (x-t), \quad \omega_0 = \frac{2\pi}{L}$$

To see $D(x-t) = \delta(x-t)$, we have to show that

$$D(x-t) = \begin{cases} 0, & x \neq t \\ \infty, & x = t \end{cases} \quad \& \int_{-\infty}^{\infty} D(x-t)\, dt = 1.$$

$$D_r(x-t) = \frac{1}{L} + \frac{2}{L} \sum_{n=1}^{\infty} r^n \cos n\omega_0 (x-t), \quad 0 < r < 1$$

$$\lim_{r\to 1^-} \underbrace{D_r(x-t)} = D(x-t).$$

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$ ✓

$$D_r(x-t) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} r^n \cos n\omega_0(x-t)$$ ✓

$$= \mathrm{Re}\left\{ \frac{2}{L}\left(\frac{1}{2} + \underline{\sum_{n=1}^{\infty} r^n e^{in\omega_0(x-t)}}\right)\right\}$$ ✓

$$= \mathrm{Re}\left\{ \frac{2}{L}\left(\frac{1}{2} + \frac{r e^{i\omega_0(x-t)}}{1-re^{i\omega_0(x-t)}}\right)\right\}$$

$$= \mathrm{Re}\left\{ \frac{2}{L}\left(\frac{1-re^{i\omega_0(x-t)}+2re^{i\omega_0(x-t)}}{2(1-re^{i\omega_0(x-t)})}\right)\right\}$$

$$= \mathrm{Re}\left\{ \frac{2}{L}\left(\frac{1+re^{i\omega_0(x-t)}}{2(1-re^{i\omega_0(x-t)})} \times \frac{1-re^{-i\omega_0(x-t)}}{1-re^{-i\omega_0(x-t)}}\right)\right\}$$

$$= \mathrm{Re}\left\{ \frac{\cancel{2}}{L}\cdot\frac{1-r^2+i2r\sin\omega_0(x-t)}{\cancel{2}\left[1+r^2-2r\cos\omega_0(x-t)\right]}\right\}$$ ✓

$$D(x-t) = \lim_{r \to 1^-} D_r(x-t) = \lim_{r \to 1^-} \frac{1}{L} \cdot \frac{1-r^2}{1+r^2-2r\cos w_0(x-t)} = \begin{cases} 0, & x \neq t \\ \infty, & x = t \end{cases}$$

$$\cos\left(\frac{2\pi(x-t)}{L}\right) \neq 1 \text{ if } x-t=0$$

$$\left[-\frac{L}{2}, \frac{L}{2}\right]$$

$$x-t \in \left[-\frac{L}{2}, \frac{L}{2}\right]$$

$$\cancel{x \in \left[t-\frac{L}{2}, t+\frac{L}{2}\right]}$$

$$x \in \left[-\frac{L}{2}, \frac{L}{2}\right]$$

To show $\displaystyle\int_{-\infty}^{\infty} D(x-t)\, dx = 1$.

$$\int_{-\infty}^{\infty} D_r(x-t)\, dx = \int_{-L/2}^{L/2} \frac{1-r^2}{L} \cdot \frac{dx}{1+r^2-2r\cos w_0(x-t)}$$

$$x-t = x'$$
$$dx = dx'$$

$$= \frac{1-r^2}{L} \int_{-L/2}^{L/2} \frac{dx}{1+r^2-2r\cos w_0 x}.$$

$$w_0 = \frac{2\pi}{L}x \qquad \frac{2\pi}{L}x = t \qquad dx = \frac{L}{2\pi}dt$$

$$= \frac{1-r^2}{\cancel{2}\,2\pi}\int_{-\pi}^{\pi}\frac{dt}{1+r^2-2r\cos t}.$$

$$= \frac{1-r^2}{\pi}\left[\int_{0}^{\pi}\frac{dt}{1+r^2-2r\cos t}\right] \quad \because \text{ integrand is even function.}$$

$$= \frac{1-r^2}{\pi}\left[\int_{0}^{\pi/2}\frac{dt}{1+r^2-2r\cos t}+\int_{\pi/2}^{\pi}\frac{dt}{1+r^2-2r\cos t}\right] \qquad t' = t-\frac{\pi}{2}$$

$$= \frac{1-r^2}{\pi}\left[\int_{0}^{\pi/2}\frac{dt}{1+r^2-2r\cos t}+\int_{0}^{\pi/2}\frac{dt}{1+r^2+2r\cos t}\right] \checkmark$$

$$= \frac{1-r^2}{\pi}\left[\int_{0}^{\pi/2}\frac{dt}{(1+r^2)^2-4r^2\frac{1}{\sec^2 t}}\right] \checkmark$$

$$= \frac{1-r^2}{\pi}\int_{0}^{\pi/2}\frac{\sec^2 t\; dt}{(1+r^2)^2(1+\tan^2 t)-4r^2}. \qquad \underline{\tan t = x}$$

$$= \frac{1-r^2}{\pi} \int_0^{\pi/2} \frac{dx}{(1+r^2)^2 - 4r^2 + (1+r^2)^2 \, x^2}$$

(Exercise)

$$= \frac{\cancel{1-r^2}}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{\cancel{1-r^2}} = 1 .$$

$$\Rightarrow \lim_{r \to 1^-} \int_{-\infty}^{\infty} D_r(x-t) \, dx = 1 .$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} D(x-t) \, dx = 1} \checkmark$$

$$\Rightarrow \quad D(x-t) = \delta(x-t).$$

$$\int_{-\infty}^{\infty} \underline{f(x)} \, \underline{\delta(x-t)} \, dx = f(t) \checkmark$$

$$f(x) = \begin{cases} f(x), & x \in [-\frac{L}{2}, \frac{L}{2}] \\ 0, & \text{otherwise} \end{cases} \checkmark$$

$$\lim_{n \to \infty} \underline{f_n(x)} = \delta(x)$$

$$\text{L.H.S} = \int_{-L/2}^{L/2} f(x)\,\delta(x-t)\,dx = \lim_{n\to\infty}\int_{-L/2}^{L/2} f_n(x-t)\cdot f(x)\,dx \qquad x - t = x'$$

$$\to f_n(x')$$

$$= \lim_{n\to\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{2^{n-1}}{L}\, \underline{\underline{f(t+x')}}\,dx'$$

$$= \lim_{n\to\infty} \frac{2^{n-1}}{L}\cdot \int_{-L/2^n}^{L/2^n} f(t+x')\,dx' \qquad t + x' = x$$

$$= \lim_{n\to\infty} \frac{2^{n-1}}{L}\cdot \int_{t-\frac{L}{2^n}}^{t+\frac{L}{2^n}} f(x)\,dx$$

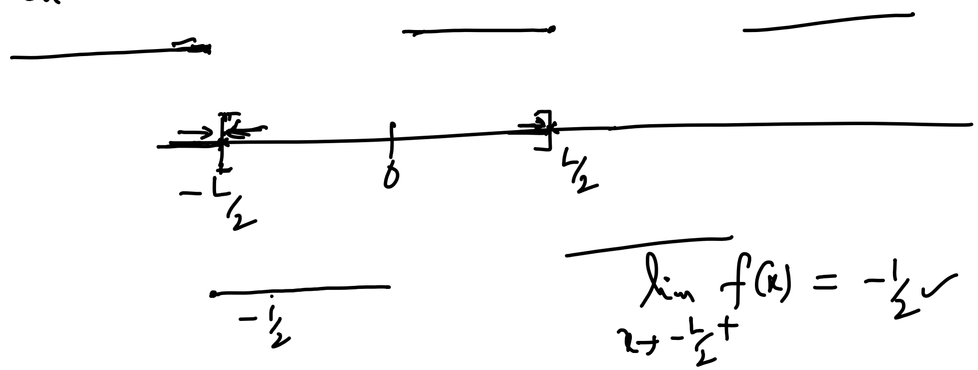
$$= \lim_{n\to\infty} \frac{\cancel{2^{n-1}}}{\cancel{L}}\cdot f(c)\,\frac{\cancel{L}}{\cancel{2^{n-1}}}, \quad t - \frac{L}{2^n} < c < t + \frac{L}{2^n}$$

$$\downarrow \qquad \downarrow$$

$$= f(t) \qquad \underline{t < c < t}$$

Example: Find the Fourier Series for the function

$$f(x) = \begin{cases} -1/2, & -\frac{L}{2} < x < 0 \\ \frac{1}{2}, & 0 < x < \frac{L}{2} \end{cases}.$$



$$\lim_{x \to -\frac{L}{2}^+} f(x) = -\frac{1}{2} \checkmark$$

$$\lim_{x \to -\frac{L}{2}^-} f(x) = \lim_{x \to \frac{L}{2}^-} f(x) = \frac{1}{2} \checkmark$$

Soln: given function is piecewise differentiable function.

$\Rightarrow$ The Fourier Series converges to $f(x)$.

ie,
$$\sum_{n=-\infty}^{\infty} c_n e^{inw_0 x} = f(x); \quad x \in \left(-\frac{L}{2}, 0\right) \cup \left(0, \frac{L}{2}\right)$$

$$w_0 = \frac{2\pi}{L}$$

Fourier coefficient
$$c_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x)\, e^{-inw_0 x}\, dx \checkmark$$

$$= \frac{1}{L}\left[-\int_{-L/2}^{0} \frac{1}{2} e^{-in\omega_0 x} dx + \int_{0}^{L/2} \frac{1}{2} e^{-in\omega_0 x} dx\right]$$

$i = \sqrt{-1}$. $\quad i^2 = -1 \Rightarrow i\,i = -1$

$-i = \frac{1}{i}$ ✓

$$= \frac{-i}{2L} \left.\frac{e^{-in\omega_0 x}}{n\omega_0}\right|_{-L/2}^{0} + \frac{i}{2L} \left.\frac{e^{-in\omega_0 x}}{n\omega_0}\right|_{0}^{L/2}$$

$$= -\frac{i}{2L}\left[\frac{1}{n\omega_0} - \frac{e^{in\omega_0 \frac{L}{2}}}{n\omega_0}\right] + \frac{i}{2L}\left[\frac{e^{-in\omega_0 \frac{L}{2}}}{n\omega_0} - \frac{1}{n\omega_0}\right]$$

$$= -\frac{i}{2Ln\omega_0} - \frac{i}{2Ln\omega_0} + \frac{i}{Ln\omega_0} \cos n\omega_0 \frac{L}{2}.$$

$$C_n = -\frac{i}{Ln\omega_0} + \frac{i}{Ln\omega_0} \cos n\omega_0 \frac{L}{2}, \quad n = \pm 1, \pm 2, ----$$

$$= \frac{i}{2\pi n}\left[-1 + (-1)^n\right], \quad n = \pm 1, \pm 2, ----$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} ✓ \\ -\dfrac{i}{\pi n}, & \text{if } \underline{n \text{ is odd}}, \end{cases} \quad n \neq 0.$$

$$c_0 = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\,dx = \frac{1}{L}\left(-\frac{1}{2}\int_{-L/2}^{0} dx + \frac{1}{2}\int_{0}^{L/2} dx\right)$$

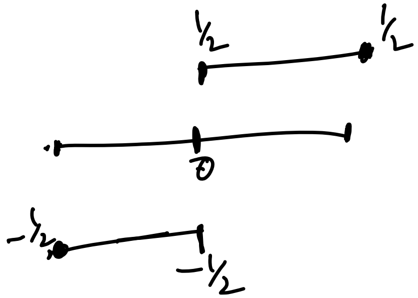
$$= \frac{-1}{2L}\left(\frac{L}{2}\right) + \frac{1}{2L}\,\frac{L}{2} = 0 ✓$$

$\Rightarrow$ Fourier Series is $\displaystyle\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = f(x)$

$$\Rightarrow \quad f(x) = \sum_{\substack{n=-\infty \\ n\neq 0,\, n \text{ is odd}}}^{\infty} \left(-\frac{i}{\pi n}\right) e^{in\omega_0 x}. ✓$$

$n \neq 0$ n is odd $\Leftrightarrow$ $n = (2k-1)$, $k = \pm 1, \pm 2, \pm 3, \ldots$

$$\Rightarrow \quad f(x) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} -\frac{i}{\pi(2k-1)} e^{i(2k-1)\omega_0 x}, \quad \underline{x \in \left(-\tfrac{L}{2}, 0\right) \cup \left(0, \tfrac{L}{2}\right)}$$



$$\Rightarrow \checkmark \; f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} -\frac{i}{\pi(2n-1)} e^{i(2n-1)\omega_0 x}; \quad x \in \left(-\tfrac{L}{2}, 0\right) \cup \left(0, \tfrac{L}{2}\right).$$

$$\Rightarrow f(x) = \frac{1}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sin(2n-1)\omega_0 x}{2n-1}, \quad x \in \left(-\tfrac{L}{2}, 0\right) \cup \left(0, \tfrac{L}{2}\right) \checkmark$$

$$\checkmark \; \boxed{0 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\cos(2n-1)\omega_0 x}{2n-1}}, \quad \omega_0 = \frac{2\pi}{L}.$$

$$\forall \; x \in \left(-\tfrac{L}{2}, 0\right) \cup \left(0, \tfrac{L}{2}\right).$$

At $x=0$:

$$f(0^+) = \frac{1}{2}$$

$$f(0^-) = -\frac{1}{2}$$

$$\Rightarrow \checkmark \frac{f(0^+)+f(0^-)}{2} = \frac{\frac{1}{2}-\frac{1}{2}}{2} = 0 \checkmark$$

$$0 = \frac{1}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sin(2n-1)w_0 x}{2n-1}\Bigg|_{x=0} = 0 \ ; \checkmark$$

At $x = \pm\frac{L}{2}$:

$$f\left(-\frac{L}{2}\right) = -\frac{1}{2}$$

$$f\left(\frac{L}{2}\right) = \frac{1}{2}$$

$$, \quad \frac{f\left(-\frac{L}{2}\right)+f\left(\frac{L}{2}\right)}{2} = 0$$

$$0 = \frac{1}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sin(2n-1)w_0 x}{2n-1}\Bigg|_{x=\pm L/2} = 0 \checkmark$$

$$\pm \sin\left\{(2n-1)\frac{2\pi}{\cancel{L}}\frac{\cancel{L}}{2}\right\} = 0$$

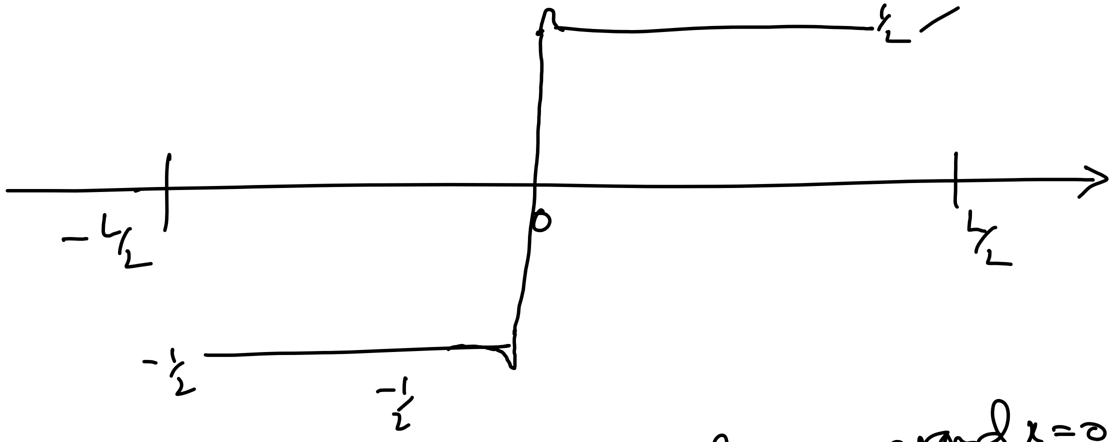
$$f(x) = \frac{1}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sin(2n-1)\omega_0 x}{2n-1}$$

$$S_k(x) = \frac{1}{\pi}\sum_{\substack{n=-k \\ n\neq 0}}^{k} \frac{\sin(2n-1)\omega_0 x}{2n-1} \longrightarrow f(x)$$

as $k \to \infty$

$k =$ Nr. fix.

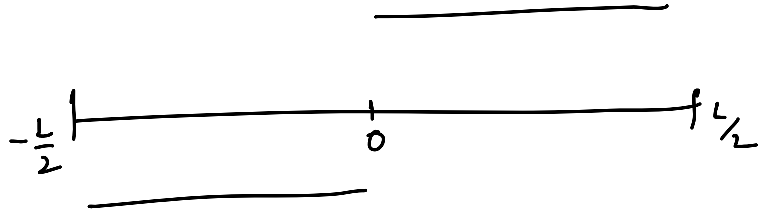
Gibb's Phenomenon:



Same % of error around $x=0$

At $x=0$

$$f(x) = \begin{cases} \frac{1}{2}, & 0 < x < \frac{L}{2} \\ -\frac{1}{2}, & -\frac{L}{2} < x < 0 \end{cases}$$



$$f(x) = \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{\infty} \left(-\frac{i}{\pi n}\right) e^{in\omega_0 x} \; ; \qquad x \in \left(-\frac{L}{2}, 0\right) \cup \left(0, \frac{L}{2}\right).$$

$-(2n-1)$

$-1, -3, -5, \text{---}$

$$= \sum_{n=1}^{\infty} -\frac{i}{\pi(2n-1)} e^{i(2n-1)\omega_0 x} + \sum_{n=1}^{\infty} \left(\frac{i}{\pi(2n-1)}\right) e^{-i(2n-1)\omega_0 x}.$$

$$= -\sum_{n=1}^{\infty} \frac{i}{\pi(2n-1)} 2i \sin(2n-1)\omega_0 x$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi(2n-1)} \sin((2n-1)\omega_0 x), \quad x \in \left(-\frac{L}{2},0\right)\cup\left(0,\frac{L}{2}\right)$$

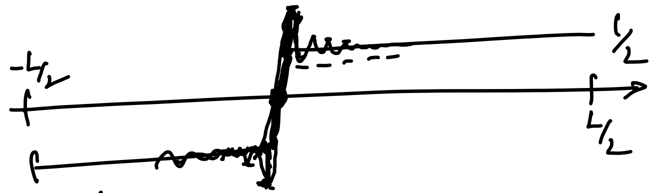
Let $S_k(x) = \sum_{n=1}^{k} \frac{2\sin((2n-1)\omega_0 x)}{\pi(2n-1)}$

As $k \to \infty$, $S_k(x) \to 0$

$$f'(x) = \sum_{n=1}^{\infty} \frac{2}{\cancel{\pi(2n-1)}} \cancel{(2n-1)} \frac{\cancel{2\pi}}{L} \cdot \cos((2n-1)\omega_0 x).$$

$$f'(x) = \frac{4}{L}\sum_{n=1}^{\infty} \cos((2n-1)\omega_0 x)$$

Since $2\cos((2n-1)\omega_0 x)\cdot\sin\omega_0 x = \sin(2n\omega_0 x) + \sin((2-2n)\omega_0 x)$,



$f'(x) = 0$

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}}$$

$$f'(x) = \sum_{n=-\infty}^{\infty} d_n e^{in\omega_0 x}, \quad d_n = \frac{1}{L}\int_{-L/2}^{L/2} f'(x)\, e^{-in\omega_0 x}\, dx$$

$$= \frac{1}{L}\cancel{\left(f(x)\, e^{-in\omega_0 x}\Big|\right)} + \frac{in\omega_0}{L}\int_{-L/2}^{L/2} f(x)\, e^{-in\omega_0 x}\, dx$$

$$\boxed{f'(x) = i\omega_0 \sum_{n=-\infty}^{\infty} n c_n e^{in\omega_0 x}}$$

$$\boxed{d_n = in\omega_0 c_n}$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{\cancel{4}^{2}}{L} \cdot \frac{\sin(2n\omega_0 x) - \sin(2(n-1)\omega_0 x)}{\cancel{2}\sin\omega_0 x}.$$

$$= \frac{2}{L\sin\omega_0 x} \sum_{n=1}^{\infty} \left[\sin(2n\omega_0 x) - \sin\left(2(n-1)\omega_0 x\right)\right].$$

$$= \frac{2}{L\sin\omega_0 x} \lim_{k\to\infty} \sum_{n=1}^{k} \left[\sin(2n\omega_0 x) - \sin\left(2(n-1)\omega_0 x\right)\right]$$

$$f'(x) = \frac{2}{L\sin\omega_0 x} \lim_{k\to\infty} \sin 2k\omega_0 x.$$

$$\underline{f'(x)} = \lim_{k\to\infty} \underline{\frac{2}{L} \cdot \frac{\sin(2k\omega_0 x)}{\sin\omega_0 x}}$$

$$\underline{S_k'(x)} = \frac{2}{L} \frac{\sin(2k\omega_0 x)}{\sin\omega_0 x}, \quad S_k(x) = \sum_{k=1}^{n} \frac{2}{\pi(2n-1)} \sin(2n-1)\omega_0 x$$

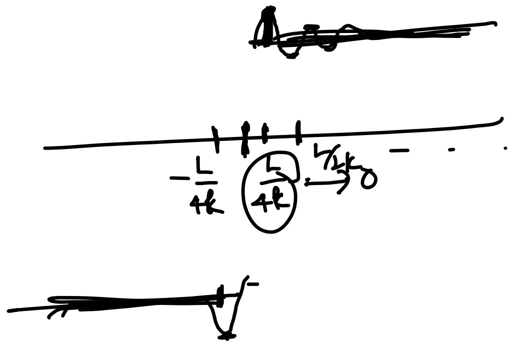
$$S_k'(x) = 0 \Rightarrow \sin(2k\omega_0 x) = 0$$

$$\Rightarrow 2k\omega_0 x = \pm n\pi, \quad n = 0, 1, 2, \dots$$

If $n=0$, $\underline{x = 0}$ ✓

If $n=1$, $x = \dfrac{\cancel{\pi}\, L}{2k \cdot 2\cancel{\pi}} = \dfrac{L}{4k}$ ✓

If $n=2$, $x = \dfrac{\cancel{2\pi}\, L}{2\cancel{\pi} k} = \dfrac{L}{2k}$ ✓

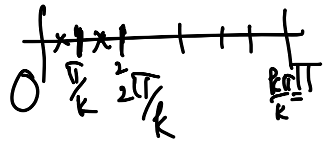


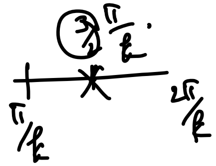
$\Rightarrow$ At $\underline{x = \pm \dfrac{L}{4k}}$, we have first local extrema on both sides of '0'.

$$S_k\left(\frac{L}{4k}\right) = \sum_{n=1}^{k} \frac{2}{\pi(2n-1)} \sin\left((2n-1)\frac{2\pi}{\cancel{L}} \cdot \frac{\cancel{L}}{\cancel{4}k}\right)$$

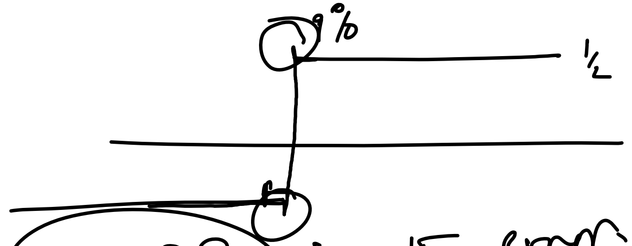
$$= \sum_{n=1}^{k} \frac{2}{\pi(2n-1)} \sin\left((2n-1)\frac{\pi}{2k}\right)$$

$$= \frac{1}{\pi}\sum_{n=1}^{k} \frac{\pi}{k} \frac{\sin\left((2n-1)\frac{\pi}{2k}\right)}{(2n-1)\frac{\pi}{2k}}$$





$$\lim_{k\to\infty} S_k\left(\pm\frac{L}{4k}\right) = \pm\frac{1}{\pi}\lim_{k\to\infty}\sum_{n=1}^{k}\frac{\pi}{k}\,\frac{\sin\left((2n-1)\frac{\pi}{2k}\right)}{(2n-1)\frac{\pi}{2k}}.$$



$$= \pm\frac{1}{\pi}\int_0^{\pi} \frac{\sin x}{x}\,dx = \pm\frac{1}{\pi}(1.852) = \pm 0.589$$

0.5

0.089 is the error

$\approx$ 9% error

for any fixed big k value.

$$\hat{f}(n) = C_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-in\omega_0 x}\,dx \quad \checkmark$$

$$\underline{\hat{f}(n)} := \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-in\omega_0 x}\, dx \iff f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{in\omega_0 x}$$

(Fourier transform) (Fourier Series or Inverse Fourier transform)

$$\underline{f(x)} \xrightarrow[\text{Periodic function}]{\text{F.T of}} \{\hat{f}(n)\}_{n\in\mathbb{Z}}$$

Properties of Fourier transform:

(1) $\overbrace{\left[\underbrace{c f_1(x) + f_2(x)}\right]}(n) = c\,\hat{f}_1(n) + \hat{f}_2(n)$ ✓ Linearity

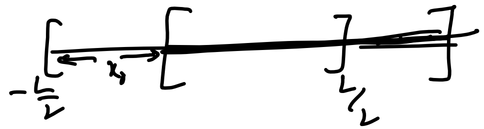
$$\text{L.H.S} = \frac{1}{L}\int_{-L/2}^{L/2} \left(c f_1(x) + f_2(x)\right) e^{-in\omega_0 x}\, dx$$

$$= c\left(\underline{\frac{1}{L}\int_{-L/2}^{L/2} f_1(x)\, e^{-in\omega_0 x}\, dx}\right) + \underline{\frac{1}{L}\int_{-L/2}^{L/2} f_2(x)\, e^{-in\omega_0 x}\, dx}$$

$$= C \cdot \hat{f}_1(n) + \hat{f}_2(n) = R.H.S.$$

$$\overline{\int_I f(x)\, g(x)\, dx}, \quad I \subset \mathbb{R}$$

$$= \int_I \overline{f(x)} \cdot \overline{g(x)}\, dx \; ✓$$



(2) $\widehat{\overline{f(x)}}(n) = \overline{\hat{f}(-n)}$ ✓ conjugation

$$L.H.S = \frac{1}{L}\int_{-L/2}^{L/2} \overline{f(x)}\, e^{-in\omega_0 x}\, dx = \frac{1}{L}\overline{\int_{-L/2}^{L/2} f(x) \cdot e^{in\omega_0 x}\, dx}$$

$$= \overline{\hat{f}(-n)} = R.H.S \; ✓$$

(3) $\widehat{f(x-x_0)}(n) = e^{-in\omega_0 x_0}\, \hat{f}(n)$. (Shift in time)

$x - x_0 = t \quad dx = dt$

$$L.H.S = \frac{1}{L}\int_{-L/2}^{L/2} f(x-x_0)\, e^{-in\omega_0 x}\, dx = \frac{1}{L}\int_{-L/2}^{L/2} f(t)\, e^{-in\omega_0 (x_0+t)}\, dt$$

$$= e^{-in\omega_0 x_0}\, \hat{f}(n) = R.H.S \; ✓$$

(4) $\widehat{f(-x)}(n) = \hat{f}(-n)$, (Time reversal)

$-x = t$ ✓

$$\text{L.H.S} = \frac{1}{L}\int_{-L/2}^{L/2} f(-x)\, e^{-in\omega_0 x}\, dx = \frac{1}{L}\int_{-L/2}^{L/2} f(t)\, e^{in\omega_0 t}\, dt$$

$$= \hat{f}(-n) = \text{R.H.S} \checkmark$$

If $f^{(n)}$ — piecewise diff fun. $\forall n$

Then f is piecewise smooth function.

✓ (5) Let $f(x)$ and $g(x)$ be two piecewise smooth periodic functions with period L.

and $h(x) = f(x) \cdot g(x)$. Then

If $\int_{-L/2}^{L/2} |h(x)|\, dx < \infty$ ✓

$$\hat{h}(n) = \widehat{f \cdot g(x)}(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\, \hat{g}(n-k) \checkmark$$

Proof: $\text{L.H.S} = \frac{1}{L}\int_{-L/2}^{L/2} \underline{f(x)\, g(x)\, e^{-in\omega_0 x}}\, dx$ ✓

$$= \frac{1}{L}\int_{-L/2}^{L/2} \underline{\sum_{k=-\infty}^{\infty} \hat{f}(k)\, e^{ik\omega_0 x}}\; g(x)\, e^{-in\omega_0 x}\, dx \checkmark$$

$$= \sum_{k=-\infty}^{\infty} \hat{f}(k)\, \underline{\frac{1}{L}\int_{-L/2}^{L/2} g(x)\, e^{-i(n-k)\omega_0 x}\, dx} \checkmark$$

$$= \sum_{k=-\infty}^{\infty} \hat{f}(k)\cdot \hat{g}(n-k) = \underline{R.H.S} \checkmark$$

$$\int \left(\sum_{k=-\infty}^{\infty} f_k(x)\right) dx \overset{?}{=} \sum_{k=-\infty}^{\infty} \int f_k(x)\, dx$$

Converges uniformly $S_n(x) = \sum_{k=-n}^{n} f_k(x) \longrightarrow \sum_{k=-\infty}^{\infty} f_k(x) = S(x)$

$S_n(x) \longrightarrow S(x)$ uniformly

$|S_n(x) - S(x)| < \epsilon, \quad n > N$

for some $N(\epsilon)$.

Convolution of two functions!

(6) $$f*g(x) := \frac{1}{L}\int_{-L/2}^{L/2} f(t)\cdot g(x-t)\, dt \checkmark$$

$f*g(t)$

then $f*g(x)$ is a also periodic. ✓

$$f * g(x+L) = \frac{1}{L}\int_{-L/2}^{L/2} f(t)\, g(x+L-t)\, dt$$

$$= \frac{1}{L}\int_{-L/2}^{L/2} f(t)\, g(x-t)\, dt$$

$$= f * g(x)$$

(7) $$\widehat{(f * g)}(n) = \hat{f}(n) \cdot \hat{g}(n)$$

$$L.H.S = \frac{1}{L}\int_{-L/2}^{L/2} f*g(x) \cdot e^{-in\omega_0 x}\, dx.$$

$$= \frac{1}{L^2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} f(t)\, g(x-t)\, dt\, e^{-in\omega_0 x}\, dx$$

$$= \frac{1}{L^2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} f(t)\, g(x_1)\, dt\, e^{-in\omega_0(x_1+t)}\, dx_1 \qquad \underline{x-t=x_1} \quad dx = dx_1$$

$$= \frac{1}{L}\int_{-L/2}^{L/2} f(t)\, e^{-in\omega_0 t}\, dt \cdot \frac{1}{L}\int_{-L/2}^{L/2} g(x_1)\, e^{-in\omega_0 x_1}\, dx_1$$

$$\underline{\widehat{f*g}(n) = \hat{f}(n)\cdot\hat{g}(n)} \checkmark$$

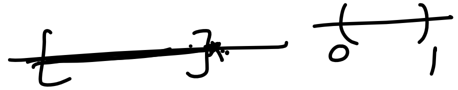
<u>Uniform Convergence</u> : $f_n(x)$, $n = 1, 2, 3, \ldots$

$x \in [a, b]$.

We say $f_n(x)$ converges $f(x)$ pointwise, if

$\lim_{n\to\infty} f_n(x) = f(x)$, for each fixed $x \in [a, b]$.

given $\epsilon > 0$, $|f_n(x) - f(x)| < \epsilon$, if $n \geq N(\epsilon, x)$ for some $N \in \mathbb{N}$.



for each $x \in [a, b]$, $N_0(\epsilon) = \sup\limits_{x \in [a,b]} \{ N(\epsilon, x) \} < \infty$. then

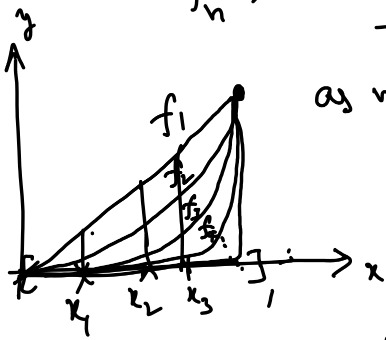
eg: $f_n(x) = \begin{cases} x^n, & x \in [0, 1) \\ 1, & x = 1 \end{cases}$

If, given $\epsilon > 0$, $|f_n(x) - f(x)| < \epsilon$, if $n \geq N_0(\epsilon)$, $\forall x \in [a, b]$.

$f_n(x) = x^n \to 0$ if $x \in [0, 1)$
$\to 1$, if $x = 1$

as $n \to \infty$

then $f_n(x)$ converges to $f(x)$ uniformly.



$f_n(x_1) \to f(x_1)$

$f_n(x) \to f(x) = \begin{cases} 0, & x \in [0, 1) \\ 1, & \text{if } x = 1 \end{cases}$

If $\lim\limits_{n \to \infty} f_n(x) = f(x)$ uniformly in $x \in [a, b]$

then, $\lim\limits_{n \to \infty} f_n'(x) = f'(x)$, $x \in (a, b)$.

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx$$ ✓

uniformly in $x \in [a,b]$.

If $$\sum_{n=1}^{\infty} f_n(x) = f(x),$$

$$S_n(x) = \sum_{k=1}^{n} f_k(x) \longrightarrow f(x) \text{ as } n\to\infty$$

(1) $$S_n'(x) \longrightarrow f'(x) = \sum_{n=1}^{\infty} f_n'(x)$$ ✓

$$\frac{d}{dx}\sum_{k=1}^{n} f_k(x) \to f'(x)$$ ✓

(2) $$\int_a^b S_n(x)\,dx \to \int_a^b f(x)\,dx.$$

$$\lim_{n\to\infty}\int_a^b \sum_{k=1}^{n} f_k(x) = \int_a^b \lim_{n\to\infty}\sum_{k=1}^{n} f_k(x)\,dx = \int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx = \int_a^b f(x)\,dx$$

* If $f(x)$ is a piecewise smooth periodic function with period 'L';
then its fourier series $\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}$ cges uniformly and absolutely.

$\sum_{n=-\infty}^{\infty} f_n(x)$ cges absolutely
if $\sum_{n=-\infty}^{\infty} |f_n(x)|$ cges pointwise

proof: M-Test: $\sum_{n=0}^{\infty} f_n(x)$ cges uniformly & absolutely

if $|f_n(x)| \le A_n$ for each 'n', $\forall x$, and $\sum_{n=0}^{\infty} A_n < \infty$.

Absolute cgence if $\sum_{n=-\infty}^{\infty} |c_n| < \infty$.

Uniform Convergence if $\left| c_n e^{in\omega_0 x} \right| \le |c_n|$, $\forall n$ & $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ $\therefore$

Cauchy - Schwartz Inequality:

Let $x, y \in \mathbb{R}$.

$$(x-y)^2 \geq 0$$

$$x^2+y^2-2xy \geq 0$$

$$\Rightarrow x^2+y^2 \geq 2xy$$

$$\Rightarrow xy \leq \frac{x^2}{2}+\frac{y^2}{2}.$$

Let $x = \dfrac{|a_n|}{\left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}}$, $y = \dfrac{|b_n|}{\left(\sum_{n=1}^{\infty} |b_n|^2\right)^{1/2}}$.

If $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ and $\sum_{n=1}^{\infty} |b_n|^2 < \infty$, then

$$\sum_{n=1}^{\infty} |a_n||b_n| \leq \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2} \cdot \left(\sum_{n=1}^{\infty} |b_n|^2\right)^{1/2} < \infty.$$

If $\sum_{n=1}^{\infty} a_n < \infty$, & $a_n > 0$, $a_1+a_2+\cdots = (a_1+a_3+\cdots)+(a_2+a_4+\cdots)$

$$\Rightarrow \quad \frac{|a_n|\,|b_n|}{\left(\sum_{n=1}^{\infty}|a_n|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}|b_n|^2\right)^{1/2}} \leq \frac{1}{2}\,\frac{|a_n|^2}{\left(\sum_{n=1}^{\infty}|a_n|^2\right)} + \frac{1}{2}\,\frac{|b_n|^2}{\left(\sum_{n=1}^{\infty}|b_n|^2\right)}$$

$$\frac{\sum_{n=1}^{\infty}|a_n|\,|b_n|}{\left(\sum_{n=1}^{\infty}|a_n|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}|b_n|^2\right)^{1/2}} \leq \frac{1}{2}\,\cancel{\frac{\sum_{n=1}^{\infty}|a_n|^2}{\left(\sum_{n=1}^{\infty}|a_n|^2\right)}} + \cancel{\frac{\sum_{n=1}^{\infty}|b_n|^2}{2\left(\sum_{n=1}^{\infty}|b_n|^2\right)}}.$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

$$\Rightarrow \quad \sum_{n=1}^{\infty}|a_n b_n| \leq \left(\sum_{n=1}^{\infty}|a_n|^2\right)^{1/2}\cdot\left(\sum_{n=1}^{\infty}|b_n|^2\right)^{1/2} \checkmark$$

claim: $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ If $f(x)$ piecewise smooth function.

$f'(x) = \sum_{n=-\infty}^{\infty} d_n e^{in\omega_0 x}$.

$f'(x) = \sum_{n=-\infty}^{\infty} in\omega_0 c_n e^{in\omega_0 x}$

$|c_n| = \left|\frac{d_n}{in\omega_0}\right| = \frac{|d_n|}{n|\omega_0|}$.

$\int_{-L/2}^{L/2} |f'|^2 dx < \infty$, $\sum_{n=-\infty}^{\infty} |d_n|^2 < \infty$

By Bessel inequality

$$\sum_{n=-\infty}^{\infty} |c_n| = |c_0| + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |c_n|$$

$$= |c_0| + \frac{1}{|\omega_0|}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{|d_n|}{n}.$$

$a_n = \frac{1}{n}$, $b_n = |d_n|$

$$\leq |c_0| + \frac{1}{|\omega_0|}\left(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^2}\right)^{1/2} \cdot \left(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |d_n|^2\right) < \infty$$

$$2\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

(7) parseval's identity

$$\frac{1}{L}\int_{-L/2}^{L/2} f(x)\,\overline{g(x)}\,dx = \sum_{n=-\infty}^{\infty} \hat{f}(n)\cdot\overline{\hat{g}(n)}.$$

proof: Let $h(x) = f(x)\cdot g(x)$.

$$\hat{h}(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\cdot\hat{g}(n-k). \checkmark$$

$$\frac{1}{L}\int_{-L/2}^{L/2} f(x)g(x)\,dx = \hat{h}(0) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\,\hat{g}(-k). \checkmark$$

If $g(x) = \overline{g(x)}$, $\hat{g}(n) = \frac{1}{L}\int_{-L/2}^{L/2} g(x)\, e^{-in\omega_0 x}\,dx$

$$= \overline{\frac{1}{L}\int_{-L/2}^{L/2} g(x)\, e^{i n w_0 x}\, dx}$$

$$\hat{g}(n) = \overline{\hat{g}(-n)}, \quad \forall\, n.$$

$$\Rightarrow \boxed{\frac{1}{L}\int_{-L/2}^{L/2} f(x)\,\overline{g(x)}\, dx = \sum_{k=-\infty}^{\infty} \hat{f}(k)\,\overline{\hat{g}(k)}}$$

If $g(x) = f(x)$, then

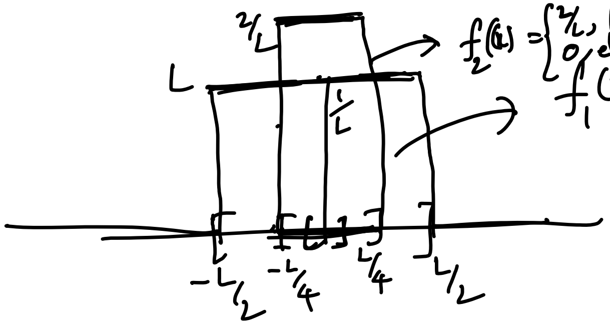
$$\frac{1}{L}\int_{-L/2}^{L/2} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2$$

$$\delta(x) = \lim_{n\to\infty} f_n(x) \checkmark$$

$$\delta(x) := \begin{cases} \infty, & x = 0 \\ 0, & \text{else.} \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx = f(0) \checkmark$$

$$\times \qquad \int_{0}^{\infty} \delta(x)\, f(x)\, dx = \frac{1}{2} f(0). \checkmark$$



$$f_2(x) = \begin{cases} 2/L, & [-L/4, L/4] \\ 0, & \text{else} \end{cases}$$

$$f_1(x) = \begin{cases} 1/L, & [-L/2, L/2] \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f_1(x)\, dx = 1 \checkmark$$

$$f_1(x) = \begin{cases} \frac{1}{L}, & \left[-\frac{L}{2}, \frac{L}{2}\right] \\ 0, & \text{else} \end{cases}$$

$$f_2(x) = \begin{cases} \frac{2}{L}, & \left[-\frac{L}{4}, \frac{L}{4}\right] \\ 0, & \text{else} \end{cases}$$

$$f_n(x) = \begin{cases} \frac{2^{n-1}}{L}, & x \in \left[-\frac{L}{2^n}, \frac{L}{2^n}\right]. \\ 0, & \text{else.} \end{cases}$$

$$\int_{-L/2^n}^{L/2^n} A\, dx = 1$$

$$A \frac{L}{2^{n-1}} = 1$$

$$\Rightarrow A = \frac{2^{n-1}}{L}$$

$$L.H.S = \int_0^\infty \lim_{n\to\infty} f_n(x)\, f(x)\, dx \qquad f_n(x) = \left[\quad \right]_{-\frac{L}{2^n}}^{\frac{L}{2^n}}$$

$$= \lim_{n\to\infty} \int_0^\infty f_n(x)\, f(x)\, dx.$$

$$= \lim_{n\to\infty} \int_0^{L/2^n} \frac{2^{n-1}}{L} f(x)\, dx \checkmark$$

$$= \lim_{n\to\infty} \frac{\cancel{2}^{n-1}}{\cancel{L}} \cdot \frac{\cancel{L}}{\cancel{2}^n} f(c), \quad 0 < c < \frac{L}{2}$$

$$= \frac{1}{2} \lim_{n\to\infty} \cdot f(c).$$

As $n\to\infty$, $L\to 0$

As $L\to 0$, $c\to 0$

$$= \frac{1}{2} f(0) \checkmark$$

$$\sum_{n=1}^{\infty} |c_n| < \infty \quad \&$$

$f(x)$ picewise continuous function

$$\longrightarrow \quad f(x), \quad \frac{f(x^+) + f(x^-)}{2}$$

$$\hat{f}(n) := \frac{2}{L} \int_{-L/2}^{L/2} f(x)\, e^{-i n \omega_0 x}\, dx$$

$$\int_{-\infty}^{0} \delta(x)\, dx = \frac{1}{2}$$

δ- function:

$$\delta(x) = \lim_{n \to \infty} f_n(x)$$

generalized function.

If $f(x)$ is even function

$$f(x) = f(-x), \quad \forall x \in \left[-\tfrac{L}{2}, \tfrac{L}{2}\right]$$

$$\left\{ \begin{array}{l} f(x) \\ \text{or} \\ \dfrac{f(x^+) + f(x^-)}{2} \end{array} \right. = \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{in\omega_0 x}, \quad \forall x \in \left[-\tfrac{L}{2}, \tfrac{L}{2}\right]$$

$$x \in \left[-\tfrac{L}{2}, \tfrac{L}{2}\right]$$

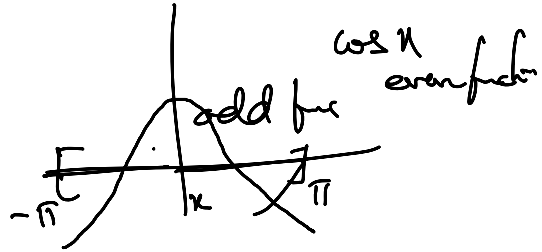
$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{in\omega_0 x} = \sum_{n=-\infty}^{0} \hat{f}(n)\, e^{in\omega_0 x} + \sum_{n=1}^{\infty} \hat{f}(n)\, e^{in\omega_0 x}$$

Fourier Series

$$\left\{ \begin{array}{l} f(x) \\ \text{or} \\ \dfrac{f(x^+) + f(x^-)}{2} \end{array} \right. = \hat{f}(0) + 2\sum_{n=1}^{\infty} \hat{f}(n) \cos n\omega_0 x \checkmark$$

$$\hat{f}(n) = \frac{1}{L} \int_{-L/2}^{L/2} f(x)\, e^{-in\omega_0 x}\, dx = \frac{1}{L} \int_{0}^{L/2} f(x)\, e^{-in\omega_0 x}\, dx + \frac{1}{L} \int_{-L/2}^{0} f(x)\, e^{-in\omega_0 x}\, dx$$

$$= \frac{1}{L}\int_0^{L/2} f(x)\, \underline{e}^{-in\omega_0 x}\, dx + \frac{1}{L}\int_0^{L/2} f(x)\, \underline{e}^{in\omega_0 x}\, dx$$



Fourier coeff:

$$\hat{f}(n) = \frac{2}{L}\cdot\int_0^{L/2} f(x)\cos n\omega_0 x\, dx \checkmark$$

$$\underline{\hat{f}(-n) = \hat{f}(n)}, \ \forall n \checkmark$$

If $f(x)$ is odd

$$f(-x) = -f(x), \ \forall x \in \left[-\frac{L}{2}, \frac{L}{2}\right] \checkmark$$

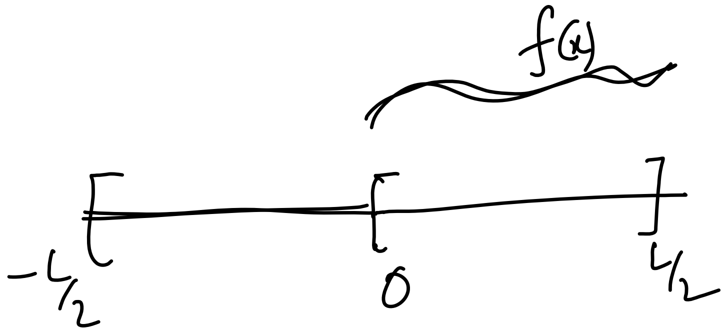
Fourier
Sine transform

$$\hat{f}(n) = \frac{2}{L}\int_0^{L/2} f(x)\sin(n\omega_0 x)\, dx \checkmark$$

Fourier Sine Series

$$2\sum_{n=1}^{\infty} \hat{f}(n)\,\sin(n\omega_0 x) = \begin{cases} f(x) \\ \text{or} \\ \dfrac{f(x^+)+f(x^-)}{2} \end{cases}$$

$$\int_{-L/2}^{L/2} f(x)\,dx = \cancel{\int_0^{L/2} f(x)\,dx} - \cancel{\int_0^{L/2} f(x)\,dx} = 0$$



even extension: $\tilde{f}_{even}(x) = \begin{cases} f(x), & x \in [0, \frac{L}{2}] \\ f(-x), & x \in [-\frac{L}{2}, 0] \end{cases}$

odd extension: $f_{odd}(x) = \begin{cases} f(x), & x \in [0, \frac{L}{2}] \\ -f(-x), & x \in [-\frac{L}{2}, 0] \end{cases}$