Note Title 30-05-2018

Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}$

$$S_n = \sum_{n=-k}^{k} C_n e^{in\omega_0 x} \xrightarrow{\text{uniformly}} \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 x}$$

If $\sum_{n=-\infty}^{\infty} |c_n| < \infty$, then $\{S_n\}$ cges uniformly.

$$|S_n(x)| \leq \sum_{n=-k}^{k} |C_n| < \sum_{n=-\infty}^{\infty} |C_n|$$

By M-test, $S_n(x)$ converges uniformly.

$f_n(x) \longrightarrow f(x)$

$|f_n(x)| \leq M_n$, for each n

If $\sum M_n < \infty$.

$\sum f_n(x)$ cges uniformly.

Theorem: If $f(x)$ is piecewise continuous function and $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ then the Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = \begin{cases} f(x), & \text{if } f \text{ is continuous at } x \\ \dfrac{f(x^+)+f(x^-)}{2}, & \text{if } f \text{ is discontinuous at `}x\text{'}. \end{cases}$$

proof: Let $S(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}$, where $c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-in\omega_0 x}\, dx$.

$$= \sum_{n=-\infty}^{\infty} \frac{1}{L}\int_{-L/2}^{L/2} f(t)\, e^{-in\omega_0 t}\, dt\; e^{in\omega_0 x}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{L}\int_{-L/2}^{L/2} f(t)\, e^{in\omega_0 (x-t)}\, dt.$$

$$= \frac{1}{L}\int_{-L/2}^{L/2} f(t)\,dt + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{L}\int_{-L/2}^{L/2} f(t)\, e^{in\omega_0(x-t)}\,dt$$

$$= \frac{1}{L}\int_{-L/2}^{L/2} f(t)\,dt + \sum_{n=1}^{\infty} \frac{2}{L}\int_{-L/2}^{L/2} f(t)\cos n\omega_0(x-t)\,dt.$$

$$= \int_{-L/2}^{L/2} f(t)\left[\frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\cos n\omega_0(x-t)\right]dt. = \int_{-L/2}^{L/2} f(t)\,\delta(x-t)\,dt = \begin{cases} f(x). & \text{at continuous point.} \\ \frac{1}{2}f(\bar{x}^{+}) + \frac{1}{2}f(\bar{x}), & \text{at discontinuous point} \end{cases}$$

Let $\quad D(x-t) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\cos n\omega_0(x-t).$

Digression: generalized function: limit of usual functions in an average sense is called a generalized function.

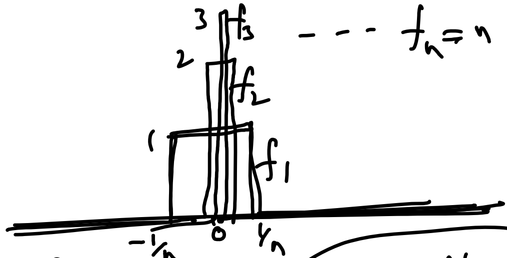
δ - function is a generalized function.

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} = \lim_{n\to\infty} f_n(x),$$

where
$$f_n(x) = \begin{cases} n/2, & -1/n < x < 1/n \\ 0, & \text{otherwise} \end{cases}$$

and
$$\int_{-\infty}^{\infty} f_n(x)\,dx = 1.$$

$$\int_{-1/n}^{1/n} \frac{n}{2}\,dx = \frac{n}{2}\cdot\frac{2}{n} = 1, \ \forall n.$$



$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \forall x \neq 0 \\ \infty, & x = 0 \end{cases} = G(x)$$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \int_{-\infty}^{\infty} G(x)\, g(x)\, dx$$

Weak convergence

$$f_n(x) \xrightarrow{\text{weakly}} G(x).$$

Property :
$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx = f(0)\,, \quad \text{for any continuous function } f(x).$$

$$\int_a^b f(x)\, dx = (b-a)\, f(c),$$
where $a < c < b$

$$L.H.S = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, \underline{f(x)}\, dx = \lim_{n\to\infty} \int_{-1/n}^{1/n} \frac{n}{2} \cdot f(x)\, dx.$$

$$= \lim_{n\to\infty} \frac{n}{2} \int_{-1/n}^{1/n} f(x)\, dx.$$

$$= \lim_{n\to\infty} \cancel{\frac{n}{2}} \cdot \cancel{\frac{2}{n}} \cdot f(c)\,, \quad -1/n < c < 1/n$$

$$= \lim_{n\to\infty} f(c)\,, \quad -1/n < c < 1/n.$$

$$= f(0) = R.H.S$$

Cor:

$$\int_{-\infty}^{\infty} \delta(x-t)\, f(x)\, dx = f(t), \quad \text{for continuous function } f(x).$$

$$\int_{0}^{\infty} \delta(x)\, f(x)\, dx = \frac{1}{2} f(0).$$

$$S(x) = \int_{-L/2}^{L/2} f(t)\, \delta(x-t)\, dt = \begin{cases} f(x). \\ \dfrac{f(x^+) + f(x^-)}{2} \end{cases}$$

Let $D(x-t) = \lim_{r \to 1^-} D_r(x-t)$, $\lim_{r\to 1^-} D_r(x-t) = \frac{2}{L}\left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n\omega_0(x-t)\right] = D(x-t)$

$-\frac{L}{2} \le x-t \le \frac{L}{2}$.

if $|r|<1$, $|r^n \cos n\omega_0(x-t)| < 1$

$$= \lim_{r\to 1^-} \mathrm{Re}\left[\frac{2}{L}\left(\frac{1}{2} + \sum_{n=1}^{\infty} r^n e^{in\omega_0(x-t)}\right)\right], \quad \omega_0 = \frac{2\pi}{L}.$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$$D_r(x-t) = \operatorname{Re}\left[\frac{\cancel{2}}{L} \cdot \frac{1 - r^2 + i\,2r \sin\left(\omega_0(x-t)\right)}{\cancel{2}\left[1 + r^2 - 2r \cos\left(\omega_0(x-t)\right)\right]}\right]$$

$$D(x-t) = \lim_{r \to 1} D_r(x-t) = \lim_{r \to 1} \frac{1}{L} \frac{1 - r^2}{1 + r^2 - 2r \cos \omega_0 (x-t)}$$

$$\cos \frac{2\pi}{L}(x-t) \neq 1 \text{ if } x \neq t.$$

$$\frac{1-r^2}{(1-r)^2} = \frac{1+r}{1-r}$$

$$= \begin{cases} 0, & \text{if } x \neq t \\ \infty, & \text{if } x = t \end{cases}$$

$$= \delta(x-t).$$

If $\int_{-\infty}^{\infty} D(x-t)\, dx = 1$, then $D(x-t) = \delta(x-t)$.

$$\int_{-\infty}^{\infty} D(x-t)\, dx = \lim_{r\to 1^-} \underline{\int_{-\infty}^{\infty} D_r(x-t)\, dx} = \lim_{r\to 1^-} 1 = \underline{1}.$$

$$\int_{-\infty}^{\infty} D_r(x-t)\, dx = \int_{-L/2}^{L/2} \frac{1-r^2}{L} \cdot \frac{1}{\left(1+r^2-2r\cos\frac{2\pi}{L}(x-t)\right)}\, dx.$$

$x-t = x'$

$$= \frac{1-r^2}{L} \int_{-L/2}^{L/2} \frac{dx}{1+r^2-2r\cos\frac{2\pi}{L}x}$$

$\frac{2\pi}{L}x = t$

$$= \frac{1-r^2}{\cancel{2}} \cdot \frac{\cancel{2}}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{1+r^2 - 2r\cos t}$$

$$= \frac{1-r^2}{\pi} \left[\int_0^{\pi/2} \frac{dt}{1+r^2-2r\cos t} + \int_{\pi/2}^{\pi} \frac{dt}{1+r^2-2r\cos t} \right]$$

$x = t - \pi/2$

$\cos t = -\cos x$

$$= \frac{1-r^2}{\pi} \cdot \left[\int_0^{\pi/2} \frac{2(1+r^2)\, dt}{(1+r^2)^2 - 4r^2 \frac{1}{\sec^2 t}} \right]$$

$$= \frac{(1-r^2)(1+r^2)2}{\pi} \int_0^{\pi/2} \frac{\sec^2 t \, dt}{(1+r^2)^2 (1+\tan^2 t) - 4r^2}$$

$\underline{\tan t = 1}$

$$= \frac{(1-r^2)(1+r^2)2}{\pi} \cdot \int_0^\infty \frac{dx}{\underline{(1+r^2)^2 - 4r^2} + (1+r^2)^2 x^2}$$

$$= \frac{\cancel{(1-r^2)}(1+r^2)2}{\pi} \cdot \frac{1}{(1-r^2)^{\cancel{2}}} \int_0^\infty \frac{dx}{1+\left(\frac{1+r^2}{1-r^2}\right)^2 x^2}$$

$$= \frac{2}{\pi} \cdot \frac{1+r^2}{1-r^2} \cdot \int_0^\infty \frac{dx}{1+\left(\frac{1+r^2}{1-r^2}x\right)^2}$$

$$= \frac{2}{\pi} \tan^{-1}\left(\frac{1+r^2}{1-r^2}x\right)\Bigg|_0^\infty = \frac{2}{\pi} \cdot \left(\frac{\pi}{2} - 0\right) = 1.$$