An introduction to coding theory

Adrish Banerjee

Department of Electrical Engineering
Indian Institute of Technology Kanpur
Kanpur, Uttar Pradesh
India

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Lecture #14B: Decoding of low density parity check codes-II: Belief Propagation Algorithm

Outline of the talk

- Message passing algorithm

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- Message passing algorithm
  - decoding in probability domain.

Probabilistic decoding

Theorem:

- Consider a sequence of m independent random variables $\mathbf{A} = [A_1, A_2, \ldots, A_m]$, where $P(A_k = 1) = p_k$. Then

$$P(\mathbf{A} \text{ has even parity}) = \frac{1}{2} + \frac{1}{2}\prod_{k=1}^{m}(1 - 2p_k)$$

and

$$P(\mathbf{A} \text{ has odd parity}) = \frac{1}{2} - \frac{1}{2}\prod_{k=1}^{m}(1 - 2p_k).$$

Probabilistic decoding

- Consider the function $\prod_{l=1}^{m}(1 - P_l + P_l t)$
- The coefficient of t^i is the probability of t i's.
- The function $\prod_{l=1}^{m}(1 - P_l - P_l t)$ is identical except for the fact that all odd powers of t are negative.
- Adding these two functions, all even powers of t double up and odd powers cancel each other.
- Letting $t = 1$, and dividing by 2 we get the probability of getting even ones.

$$P(\mathbf{A} \text{ has even parity}) = \frac{1}{2} + \frac{1}{2}\prod_{k=1}^{m}(1 - 2p_k)$$

- Similarly we can prove

$$P(\mathbf{A} \text{ has odd parity}) = \frac{1}{2} - \frac{1}{2}\prod_{k=1}^{m}(1 - 2p_k).$$

Notation

- Consider the code with parity check matrix, $\mathbf{H}$:

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

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- $C_i = \{j : h_{j,i} = 1\}$ = location of 1's in column i of H = the parity checks involving the i^{th} codebit.

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- $p_{k,j}(i) = P(c_{k,j}(i) = 1|y_{k,j}(i))$.

Theorem

- The *a posteriori* probability (APP) ratio for c_i given the received word $\mathbf{y} = [y_0, y_1, \ldots, y_{n-1}]$ and given the event $S_i = \{$ the bits in $\mathbf{c}$ satisfy the parity check constraints involving $c_i\}$, is given by

$$\frac{P(c_i = 0|\mathbf{y}, S_i)}{P(c_i = 1|\mathbf{y}, S_i)} = \frac{(1 - p_i)}{p_i} \frac{\prod_{j \in C_i} \left(1 + \prod_{i' \in R_{j \setminus i}} (1 - 2p_{i'j(i)})\right)}{\prod_{j \in C_i} \left(1 - \prod_{i' \in R_{j \setminus i}} (1 - 2p_{i'j}(i))\right)}$$

Proof

- From Bayes' rule:

$$\frac{P(c_i = 0|\mathbf{y}, S_i)}{P(c_i = 1|\mathbf{y}, S_i)} = \frac{\overbrace{P(c_i = 0|y_i)}^{1-p_i} P(S_i|c_i = 0, \mathbf{y})}{\underbrace{P(c_i = 1|y_i)}_{p_i} P(S_i|c_i = 1, \mathbf{y})}.$$

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- Let's consider the term $P(S_i|c_i = 0, \mathbf{y})$. Given $c_i = 0$, S_i holds if each of w_c parity checks involving c_i has the property that the $w_r - 1$ bits in the check *other than* c_i have even parity.

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- Let's consider the term $P(S_i|c_i = 0, \mathbf{y})$. Given $c_i = 0$, S_i holds if each of w_c parity checks involving c_i has the property that the $w_r - 1$ bits in the check *other than* c_i have even parity.
- For parity check $j \in C_i$, the probability that the $w_r - 1$ bits other than c_i have even parity is given by the lemma to be:

$$\frac{1}{2} + \frac{1}{2} \prod_{i' \in R_{j \setminus i}} (1 - 2p_{i',j}(i)).$$

Probabilistic Decoding

- The independence of the y_i's means that the probability that *all* w_c parity checks involving c_i are satisfied (given $c_i = 0$) is just

$$P(S_i|c_i = 0, \mathbf{y}) = \prod_{j \in C_i} \left(\frac{1}{2} + \frac{1}{2} \prod_{i' \in R_{j \setminus i}} (1 - 2p_{i'j}(i)) \right).$$

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- Similar analysis assuming $c_i = 1$ yields

$$P(S_i|c_i = 1, \mathbf{y}) = \prod_{j \in C_i} \left(\frac{1}{2} - \frac{1}{2} \prod_{i' \in R_{j \setminus i}} (1 - 2p_{i'j}(i)) \right).$$

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- $r_{j,i}(x)$ is the message passed from the j^{th} check node to the bit node $X_i = x$.

$$\begin{aligned} r_{j,i}(+1) &= P(\text{Parity check } j \text{ satisfied} | c_i = 0, \text{other bits} \\ &\qquad \text{in check } j \text{ have distributions given by } q) \\ &= \frac{1}{2} + \frac{1}{2} \prod_{i' \in R_{j \setminus i}} (1 - 2q_{i',j}(-1)). \end{aligned}$$

and so

$$\begin{aligned} r_{j,i}(-1) &= P(\text{Parity check } j \text{ satisfied} | c_i = 1, \text{other bits} \\ &\qquad \text{in check } j \text{ have distributions given by } q) \\ &= P(\text{Parity check } j \text{ not satisfied} | c_i = 0, \text{other bits} \\ &\qquad \text{in check } j \text{ have distributions given by } q) \\ &= 1 - r_{j,i}(+1). \end{aligned}$$

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$$\begin{aligned} q_{i,j}(+1) &= P(X_i = +1 | y_i, \text{ information from check nodes} \\ &\qquad \text{other than } j^{th} \text{ check node}). \end{aligned}$$

$$\frac{q_{i,j}(+1)}{q_{i,j}(-1)} = \frac{(1 - p_i)}{p_i} \frac{\prod_{j' \in C_{i \setminus j}} r_{j',i}(+1)}{\prod_{j' \in C_{i \setminus j}} r_{j',i}(-1)}.$$

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For all i, j such that $h_{j,i} = 1$. (So i indexes the bit nodes and j indexes the parity checks.)

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Probabilistic Decoding

- **Step 4:** Compute the hard decisions and decide if it's time to stop.

$$\hat{c}_i = \begin{cases} 1 & \text{if } Q_i(-1) \geq 0.5; \\ 0 & \text{otherwise.} \end{cases}$$

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- If($([\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_{n-1}]H^T = 0)$ or (Maximum # of iterations reached) then stop, else repeat Steps 1-4.

Example

C_0	C_1	C_2
C_3	C_4	C_5
C_6	C_7	

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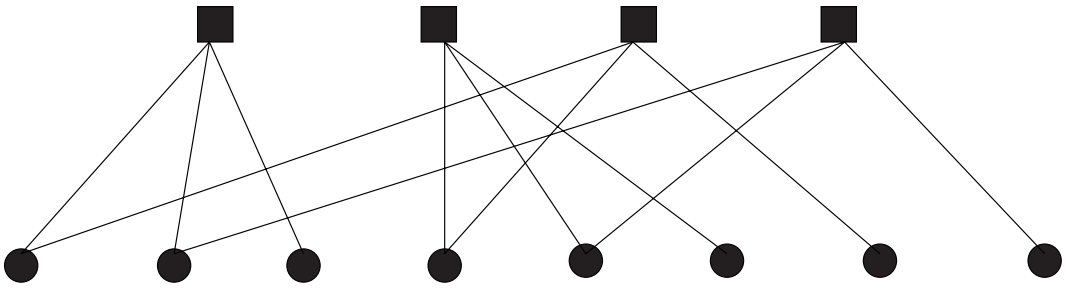
- $n = 8$, $m = n - k = 4$, $d_{\min} = 3$

Example



Example

1	0	1
0	1	1
1	1	

modulation →

−1	+1	−1
+1	−1	−1
−1	−1	

↓ transmission, $\sigma^2 = 0.5$

+0.2	+0.2	−0.9
+0.6	+0.5	−1.1
−0.4	−1.2	

Example

Initialization:

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- $q_{1,0}(-1) = q_{1,3}(-1) = 0.310$ and $q_{1,0}(+1) = q_{1,3}(+1) = 0.690$.

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- $q_{2,0}(-1) = 0.973$ and $q_{2,0}(+1) = 0.027$.

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- $q_{2,0}(-1) = 0.973$ and $q_{2,0}(+1) = 0.027$.
- $q_{3,1}(-1) = q_{3,2}(-1) = 0.083$ and $q_{3,1}(+1) = q_{3,1}(+1) = 0.917$.

Example

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- $q_{5,1}(-1) = 0.988$ and $q_{5,1}(+1) = 0.012$.
- $q_{6,2}(-1) = 0.832$ and $q_{6,2}(+1) = 0.168$.

Example

- $q_{4,1}(-1) = q_{4,3}(-1) = 0.119$ and $q_{4,1}(+1) = q_{4,3}(+1) = 0.881$.
- $q_{5,1}(-1) = 0.988$ and $q_{5,1}(+1) = 0.012$.
- $q_{6,2}(-1) = 0.832$ and $q_{6,2}(+1) = 0.168$.
- $q_{7,3}(-1) = 0.992$ and $q_{7,3}(+1) = 0.008$.

Example

- Now compute $r_{j,i}$'s from $q_{i,j}$'s:

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Adrish Banerjee Department of Electrical Engineering Indian Institute of Technology Kanpur Kanpur, Uttar Pradesh India
An introduction to coding theory

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  - $r_{2,0}(+1) = 0.5 + 0.5(1 - 2(0.083))(1 - 2(0.832)) = 0.223.$

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- $r_{j,i}(-1) = 1 - r_{j,i}(+1).$

Example

Now compute $q_{i,j}$'s from $r_{j,i}$'s:

$$\begin{aligned} \tilde{q}_{0,0}(+1) &= (1 - p_0) \prod_{j' \in C_{0\setminus 0}} r_{j',0}(+1) \\ &= (0.69) r_{2,0}(+1) \\ &= (0.69)(0.223) = 0.154. \end{aligned}$$

and

$$\begin{aligned} \tilde{q}_{0,0}(-1) &= p_0 \prod_{j' \in C_{0\setminus 0}} r_{j',0}(-1) \\ &= 0.31 r_{2,0}(-1) \\ &= 0.31(0.777) = 0.241. \end{aligned}$$

Example

- This means

$$q_{0,0}(+1) = \frac{0.154}{0.154 + 0.241} = 0.39$$

and

$$q_{0,0}(-1) = \frac{0.241}{0.154 + 0.241} = 0.61.$$

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- Finally, compute the APP's:
- Note: $\tilde{Q}_i(+1) = \tilde{q}_{i,j}(+1) \cdot r_{j,i}(+1)$, which means

$$\tilde{Q}_0(+1) = \tilde{q}_{0,0}(+1) \cdot r_{0,0}(+1) = 0.154 \cdot 0.32 = 0.0493$$

and

$$\tilde{Q}_0(-1) = \tilde{q}_{0,0}(-1) \cdot r_{0,0}(-1) = 0.241 \cdot 0.68 = 0.164.$$

Example

- This yields the APP

$$Q_0(+1) = \frac{0.0493}{0.0493 + 0.164} = 0.23$$

and

$$Q_0(-1) = \frac{0.164}{0.0493 + 0.164} = 0.77.$$

Example

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- The other Q_i's can be computed similarly.

Probabilistic Decoding

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- Average number of iterations required to decode is bounded by a quantity proportional to the log of the log of the block length.