An introduction to coding theory

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Lecture #13: Low density parity check codes

Outline of the lecture

- Introduction

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- Tanner graphs

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- Construction of regular LDPC codes

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- Irregular LDPC codes
- Random construction of irregular LDPC codes.

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- A vector $\mathbf{v}$ is **very sparse** if its density vanishes as its length increases.
- The **overlap** between two vectors is the number of 1's in common between them.

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  - Number of ones in the parity check matrix $\mathbf{H} = w_c \cdot n = w_r \cdot m$.
  - $m \geq n - k \implies R = k/n \geq 1 - (w_c/w_r)$, and thus $w_c < w_r$.

Regular low-density parity check code

1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0
0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0
0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0
0	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1
1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0
0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0
0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0
0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0
0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1

Example of a regular low density code matrix; $n = 20$, $w_c = 3$, $w_r = 4$

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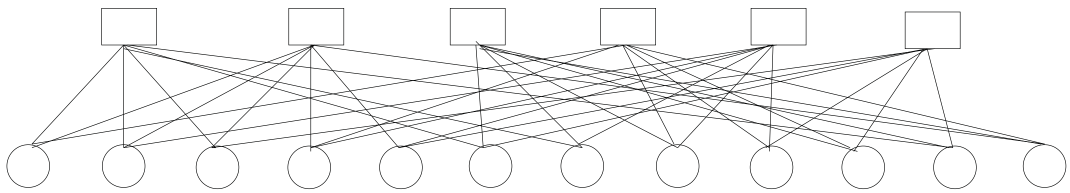
  - One class of nodes is the "variable nodes" corresponding to n bits in the codeword.
  - Second class of nodes is "check nodes" corresponding to m parity check equations.
  - An edge connects a variable node to the check node if and only if that particular bit is included in the parity check equation.

Tanner Graphs

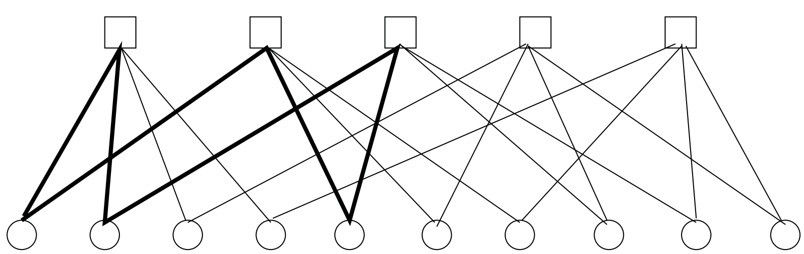
- Example of a regular low density code matrix; $n = 12$, $w_c = 3$, $w_r = 6$

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$



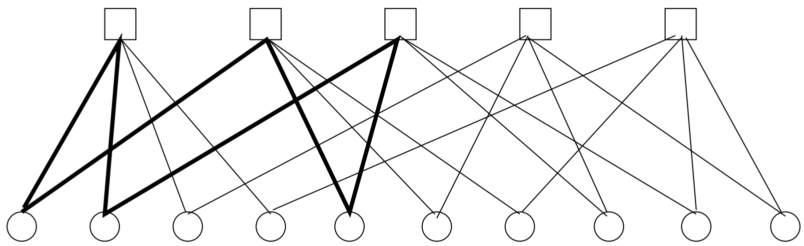
Tanner Graphs

- A cycle of length l in a Tanner graph is a path comprised of l edges from a node back to the same node.



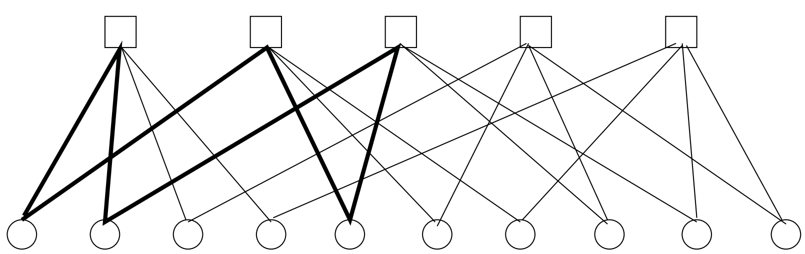
Tanner Graphs

- A cycle of length l in a Tanner graph is a path comprised of l edges from a node back to the same node.
- Example: The bipartite graph has a cycle of length six.



Tanner Graphs

$$\mathbf{H} = \begin{bmatrix} \boxed{1} & \boxed{1} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \boxed{1} & 0 & 0 & 0 & \boxed{1} & 1 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & \boxed{1} & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$



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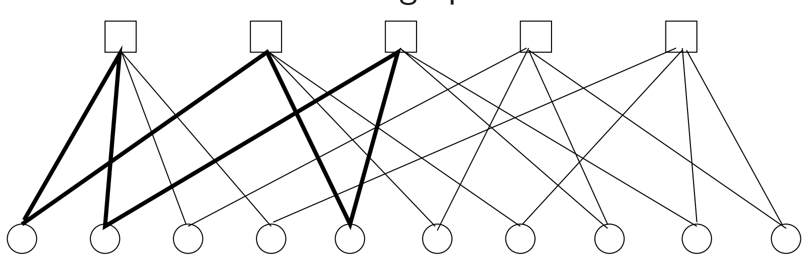

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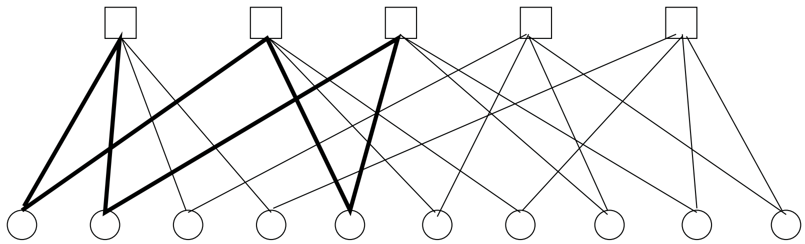

- The girth of this Tanner graph is six.

Gallager's construction for regular (n, w_c, w_r) code

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- The other sub-matrices are merely column permutations of the first sub-matrix.

Gallager's construction for regular (n, w_c, w_r) code

1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0
0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0
0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0
0	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1
1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0
0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0
0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0
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Example of a regular low density code matrix; $n = 20$**,** $w_c = 3$**,** $w_r = 4$

MacKay's construction

- An m by n matrix (m rows, n columns) is created at random with weight per column w_c, and weight per row w_r, and overlap between any two columns no greater than 1.

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- Another way of constructing regular LDPC codes is to build the parity check matrix from non-overlapping random permutation matrices.

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Construction of low density parity check codes

- A permutation matrix is just the identity matrix with its row re-ordered, e.g.

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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- A circulant matrix is defined by the property that each row is a cyclic shift of the previous row to the right by one position.

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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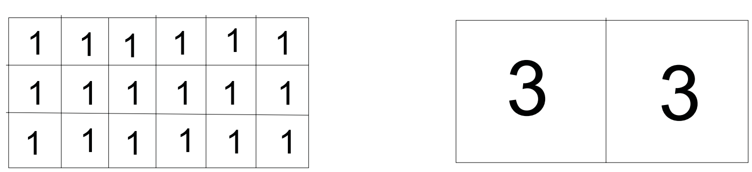

Schematic Illustration of Regular Gallager Codes

Notation: An integer represents a number of permutation matrices superposed on the surrounding square.

Column Weight	Fraction of columns	Row weight	Fraction
3	1	6	1

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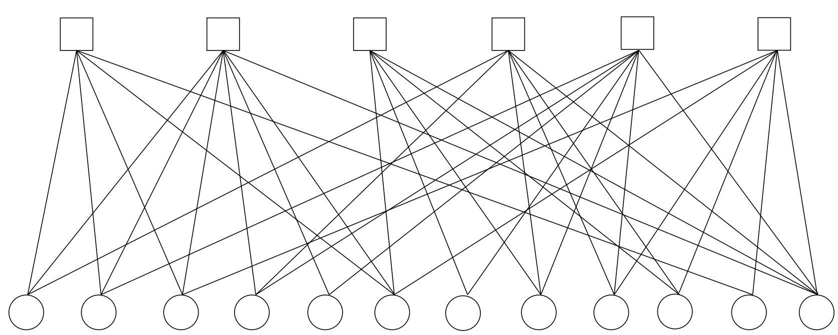
- A degree distribution $\gamma(x) = \sum_i \gamma_i x^{i-1}$ is simply a polynomial with nonnegative real coefficients satisfying $\gamma(1) = 1$.
- An irregular low-density code is a code of block-length N with a sparse parity check matrix where column distribution $\lambda(x)$ and row distribution $\rho(x)$ is respectively given by

$$\lambda(x) = \sum_{i>1} \lambda_i x^{i-1}$$

$$\rho(x) = \sum_{i>1} \rho_i x^{i-1}$$

where λ_i and ρ_i denote the fraction of edges incident to variable and check nodes with degree i, respectively.

Irregular Low-density parity check code



$$\lambda(x) = \tfrac{1}{4}x + \tfrac{7}{12}x^2 + \tfrac{1}{12}x^3 + \tfrac{1}{12}x^4$$
$$\rho(x) = \tfrac{1}{3}x^4 + \tfrac{1}{3}x^5 + \tfrac{1}{3}x^6$$

Irregular low density parity check codes

Construction of irregular LDPC code

Step 1 : Selecting a profile that describes the desired number of columns of each weight and the desired number of rows of each weight.

Step 2 : Construction method, i.e. algorithm for putting edges between the vertices in a way that satisfies the constraints.

Random construction of Irregular LDPC Codes

The edges are placed “completely at random” subject to the profile constraints. One way of implementing it is shown below.

- Make a list of all columns in the matrix, with each column appearing in the list a number of times equal to its weight.

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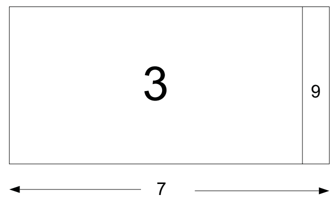
- Map one list onto the other by a random permutation, taking care not to create duplicate entries.

Construction of Irregular LDPC Codes



Notation: integers "3" and "9" represent the column weights.

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3 9	11/12 1/12	7	1

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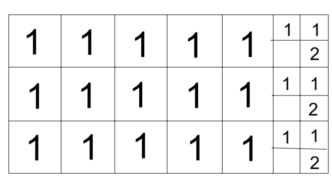

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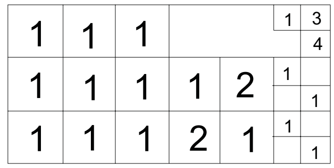

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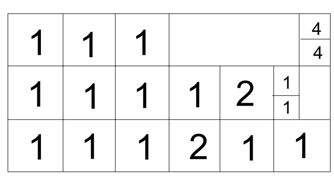

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