An introduction to coding theory

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Lecture #1C: Introduction to error control coding-III

Outline of the lecture

- Types of Codes

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- Decoding strategies

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- Decoding strategies
- Error control strategies

Types of Channel Codes

Block codes:

- The information sequence is partitions into message blocks of k-information bits each, represented as:

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- The encoder for a block code is memoryless.

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- The codeword sequence, in general, can be non-binary, but we only consider binary codes since they are the most commonly used in practice.

Block Codes

- *Example*: Let $k = 3$ and $n = 6$. The following table gives a block code of length 6. The code rate is $R = \frac{1}{2}$.

Message	Codewords
(0 0 0)	(0 0 0 0 0 0)
(1 0 0)	(0 1 1 1 0 0)
(0 1 0)	(1 0 1 0 1 0)
(1 1 0)	(1 1 0 1 1 0)
(0 0 1)	(1 1 0 0 0 1)
(1 0 1)	(1 0 1 1 0 1)
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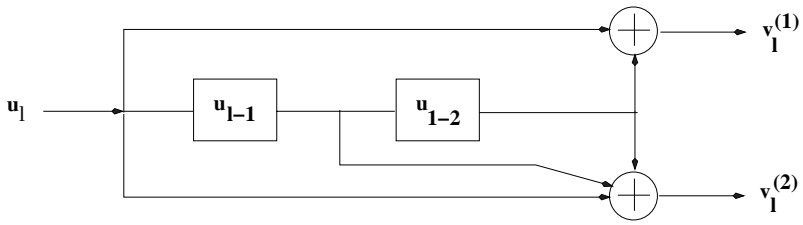
- The set of sequences produced by a k-input, n-output encoder of memory order m is called an (n, k, m) convolutional code.
- The values of n and k are much smaller for convolutional codes compared to the block codes.

Convolutional Codes



- Let $k = 1$, $n = 2$ and $m = 2$. The following circuit generates a $(2, 1, 2)$ convolutional code.

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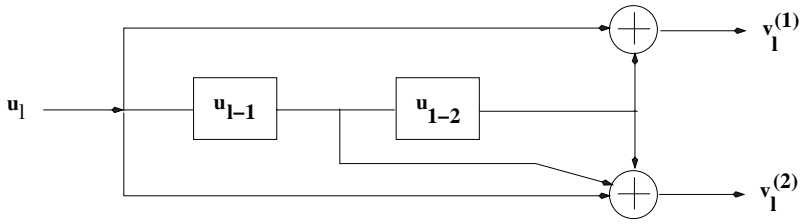

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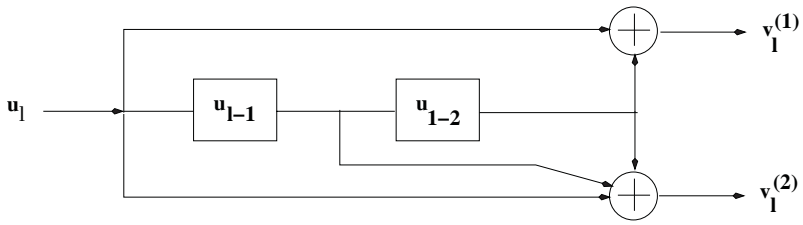

- Let $k = 1$, $n = 2$ and $m = 2$. The following circuit generates a $(2, 1, 2)$ convolutional code.
- Input: u_l
- Outputs:

$$\begin{aligned} v_l^{(1)} &= u_l + u_{l-2} \\ v_l^{(2)} &= u_l + u_{l-1} + u_{l-2} \end{aligned}$$

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- A *decoding rule* is an assignment of an estimate $\hat{\mathbf{v}}$ to each of the received sequence $\mathbf{r}$.

Decoding Strategies

- The average probability of error is given by

$$\begin{aligned} P(E) = P(\hat{\mathbf{v}} \neq \mathbf{v}) &= \sum_{\mathbf{r}} P(E/\mathbf{r})P(\mathbf{r}) \\ &= \sum_{\mathbf{r}} P(\hat{\mathbf{v}} \neq \mathbf{v}/\mathbf{r})P(\mathbf{r}) \end{aligned}$$

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- Minimizing $P(\hat{\mathbf{v}} \neq \mathbf{v}/\mathbf{r})$ is equivalent to maximizing $P(\hat{\mathbf{v}} = \mathbf{v}/\mathbf{r})$.

Decoding Strategies

- For each $\mathbf{r}$, compute

$$P(\mathbf{v}/\mathbf{r}) = \frac{P(\mathbf{r}/\mathbf{v})P(\mathbf{v})}{P(\mathbf{r})} \quad \text{Bayes' rule}$$

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- A Maximum a-posteriori probability (MAP) decoder chooses $\hat{v}$ such that $P(\mathbf{v}/\mathbf{r})$ is maximized.

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- Since $\log x$ is a monotone increasing function of x, maximizing $P(\mathbf{r}/\mathbf{v})$ is equivalent to maximizing $\log P(\mathbf{r}/\mathbf{v})$.

Decoding Strategies

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- Hamming distance $d(\mathbf{r}, \mathbf{v})$ between $\mathbf{r}$ and $\mathbf{v}$ is the number of positions for which $r_i \neq v_i$.

$$\log P(\mathbf{r}/\mathbf{v}) = d(\mathbf{r}, \mathbf{v}) \log \left(\frac{p}{1-p} \right) + n \log(1-p)$$

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- Note $\log \left(\frac{p}{1-p} \right) < 0$ for $p < 1/2$ and $n \log(1-p)$ is a constant.
- For each $\mathbf{r}$, choose $\hat{\mathbf{v}}$ as the codeword $\mathbf{v}$ which minimizes the Hamming distance $d(\mathbf{r}, \mathbf{v})$.

Error control strategies

Forward Error Correction (FEC):

- In *one-way* system, transmission takes place only in one direction, i.e. from transmitter to receiver.

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- The error correcting codes used in such a system are referred as *forward error correction (FEC)* codes.

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Automatic Repeat reQuest (ARQ):

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- In *two-way* system, there exist a feedback path from the receiver to the transmitter.
- Error correction can be achieved for two-way system using error detection and retransmission, also known as *automatic repeat request (ARQ)*.

Error control strategies

Hybrid Automatic Repeat reQuest (HARQ):

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- For channels with feedback, one can use ARQ protocols in combination with FEC to improve system performance. These types of schemes are known as hybrid-ARQ (HARQ) schemes.
- In HARQ protocols, the transmitted data is encoded for both error correction and error detection. If the receiver detects an error after decoding, it sends a negative acknowledgment (NACK) to the transmitter, which then retransmits the data.