An introduction to coding theory

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Lecture #9B: Convolutional codes: Distance properties

Outline of the lecture

- Weight enumerating function calculation

Distance properties

- The minimum free distance of a convolutional code

$$d_{\text{free}} \stackrel{\Delta}{=} \min_{\mathbf{u},\mathbf{u}'} d(\mathbf{v},\mathbf{v}') : \mathbf{u} \neq \mathbf{u}'$$

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- d_{free} is also the minimum weight non-zero sequence, i.e.

$$d_{\text{free}} = \min\{w(\mathbf{v}) : \mathbf{u} \neq 0\}$$

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  - The weight of a code sequence is the power of X in the path gain of the corresponding path.

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- The weight distribution function of a convolutional code is obtained by applying Mason's gain formula to compute the generating function

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- Let $\{i,j,k\}$ denote the set of all triples of non-touching loops, and so on., then define

$$\Delta = 1 - \sum_{\{i\}} C_i + \sum_{\{i,j\}} C_i C_j - \sum_{\{i,j,k\}} C_i C_j C_k + \cdots$$

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- Let G_i be the graph obtained by removing all the states on the i^{th} forward path and all the branches connected to these states, and let Δ_i be defined similarly as Δ for the graph G_i.

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- The augmented transfer function is given by

$$T(X, Y, Z) = \sum_{i,j,l} A_{i,j,l} X^i Y^j Z^l$$

where $A_{i,j,l}$ is the number of code sequences of weight i, whose corresponding information sequence has weight j, and which has length l branches.

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$$T(X) = T(X, Y)|_{Y=1} = T(X, Y, Z)|_{Y=Z=1}$$
- Conditional weight enumerating function (CWEF): Enumerates weights of all codewords associated with particular information weights.

Weight enumerating function calculation

- For information weight of j, the CWEF is obtained as
$$T_j(X) = \sum_i A_{i,j} X^i$$
where $A_{i,j}$ represents number of code sequences with weight i, and information weight j.

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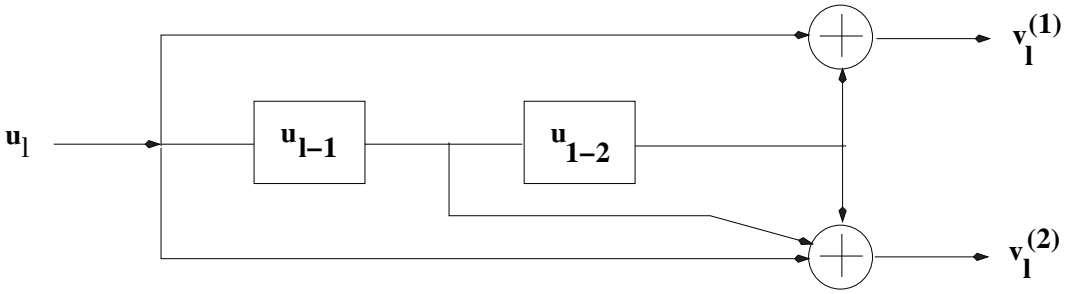
- Input-redundancy weight enumerating function (IRWEF): It is defined for systematic encoders as follows

$$T(W, Y, Z) = \sum_{w,j,l} A_{w,j,l} W^w Y^j Z^l$$

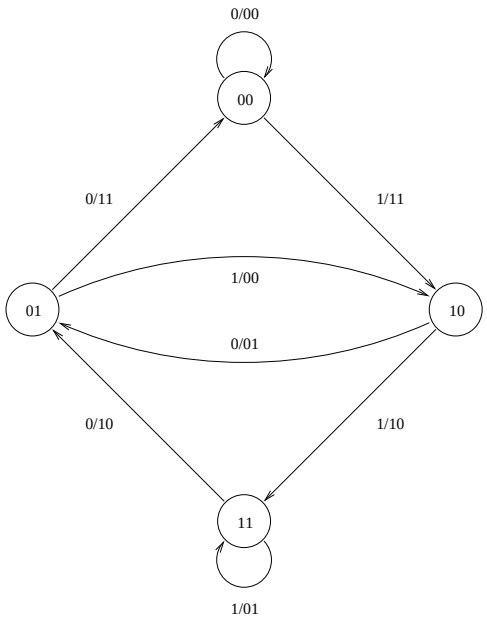
where $A_{w,j,l}$ is the number of code sequences of length l with information weight j, and parity weight w.

Convolutional encoder



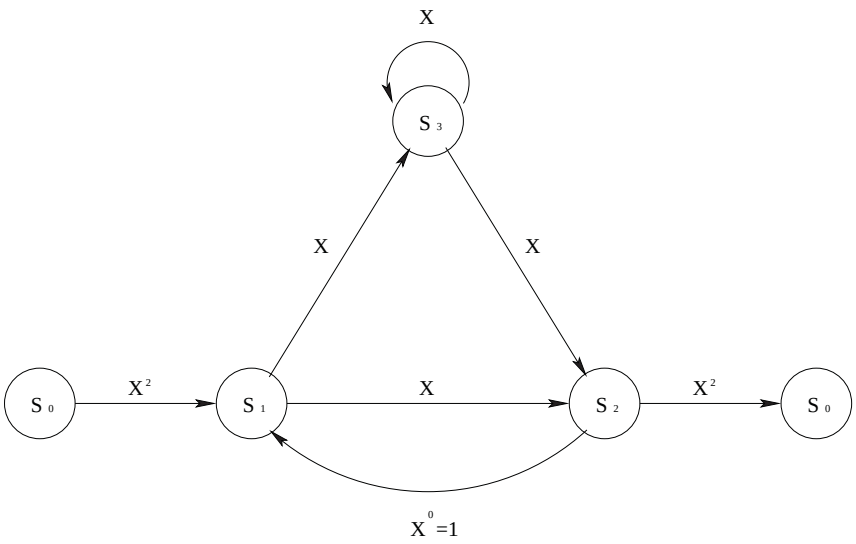
$(2, 1, 2)$ convolutional code

State diagram



State diagram of $(2, 1, 2)$ convolutional code

Modified state diagram



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  - $\{1, 2\}$ with $C_1C_2 = X^2$.
- No triples of non-touching loops.
- Hence,

$$\begin{aligned}\Delta &= 1 - (C_1 + C_2 + C_3) + C_1C_2 \\ &= 1 - 2X - X^2 + X^2 \\ &= 1 - 2X\end{aligned}$$

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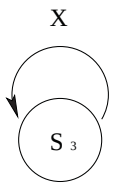

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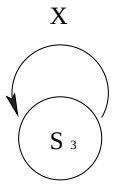

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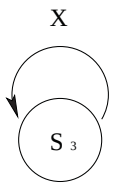

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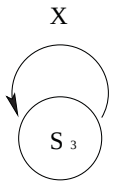

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- The graph G_2 obtained by removing the states on the forward path 2 is empty.
- Hence $\Delta_2 = 1$.

Weight enumerating function calculation

- The transfer function $T(X)$ is given by

$$\begin{aligned} T(X) &= \frac{F_1\Delta_1 + F_2\Delta_2}{\Delta} \\ &= \frac{X^5(1-X) + X^6 \cdot 1}{1-2X} \\ &= \frac{X^5}{1-2X} \\ &= X^5 + 2X^6 + 4X^7 + \cdots + 2^k X^{k+5} + \cdots \end{aligned}$$

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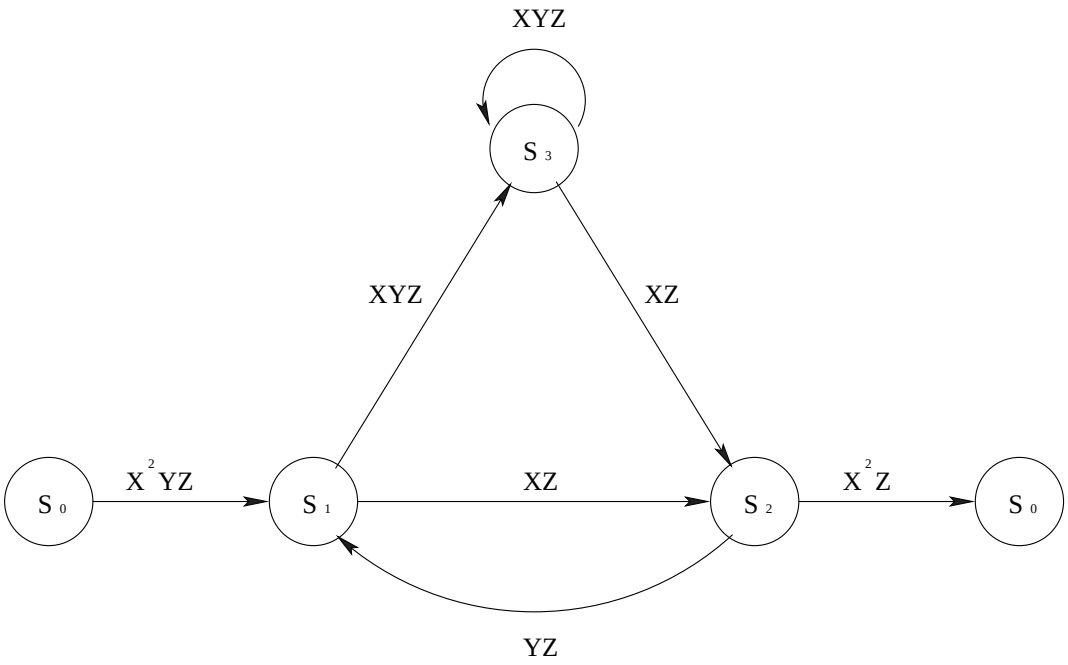
- $d_{\text{free}} = 5$

Augmented state diagram



Input-output weight enumerating function calculation

- The transfer function $T(X, Y, Z)$ is given by

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