An introduction to coding theory

Adrish Banerjee

Department of Electrical Engineering
Indian Institute of Technology Kanpur
Kanpur, Uttar Pradesh
India

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Lecture #12: Performance Bounds for Convolutional Codes

Introduction

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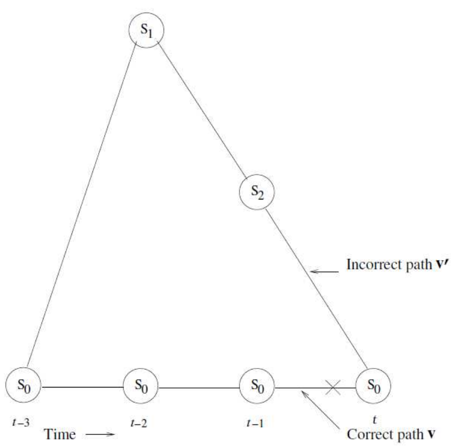
$$\mathbf{G}(\mathbf{D}) = [1 + D \quad 1 + D^2 \quad 1 + D + D^2]$$

- The Input Output Weight Enumerating Function (IOWEF) of this encoder is given by

$$\begin{aligned} A(W, X, L) &= \frac{X^7 W L^3}{1 - XWL(1 + X^2 L)} \\ &= X^7 W L^3 + X^8 W^2 L^4 + X^9 W^3 L^5 + X^{10}(W^2 L^5 + W^4 L^6 + \cdot \end{aligned}$$

First Error Event

- A first event error happens at an artibrary time t if the all zero path is eliminated for the first time in favor of an incorrect path.



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- So, it must be one of the path enumerated by the codeword weight enumerating function.
- Assuming that the incorrect path has weight d, a first event error happens with probability

$$P_d = \begin{cases} \sum_{e=\frac{d+1}{2}}^{d} \begin{pmatrix} d \\ e \end{pmatrix} p^e(1-p)^{d-e} & \text{odd} \\ \frac{1}{2} \begin{pmatrix} d \\ d/2 \end{pmatrix} p^{d/2}(1-p)^{d/2} + \sum_{e=\frac{d}{2}+1}^{d} \begin{pmatrix} d \\ e \end{pmatrix} p^e(1-p)^{d-e} & \text{even} \end{cases}$$

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- Thus the first event error probability at time t can be bounded using union bound by the sum of the error probabilities of each of these paths.
- If all incorrect paths of length greater than t are also included, then the first event error probability at any time t can be bounded by

$$P_f(E) < \sum_{d=d_{\text{free}}}^{\infty} A_d P_d$$

where A_d is the number of codewords of weight d.

First Error Event

- For odd d, we can write

$$\begin{aligned} P_d &= \sum_{e=\frac{d+1}{2}}^{d} \binom{d}{e} p^e(1-p)^{d-e} \\ &< \sum_{e=\frac{d+1}{2}}^{d} \binom{d}{e} p^{d/2}(1-p)^{d/2} \\ &= p^{d/2}(1-p)^{d/2} \sum_{e=\frac{d+1}{2}}^{d} \binom{d}{e} \\ &< p^{d/2}(1-p)^{d/2} \sum_{e=0}^{d} \binom{d}{e} \\ &= 2^d p^{d/2}(1-p)^{d/2} \end{aligned}$$

First Error Event

- Similarly, for even d, we have

$$\begin{aligned} P_d &= \frac{1}{2}\binom{d}{d/2} p^{d/2}(1-p)^{d/2} + \sum_{e=(d/2)+1}^{d} \binom{d}{e} p^e(1-p)^{d-e} \\ &< \sum_{e=(d/2)}^{d} \binom{d}{e} p^e(1-p)^{d-e} \\ &< \sum_{e=(d/2)}^{d} \binom{d}{e} p^{d/2}(1-p)^{d/2} \\ &= p^{d/2}(1-p)^{d/2} \sum_{e=(d/2)}^{d} \binom{d}{e} \\ &< p^{d/2}(1-p)^{d/2} \sum_{e=0}^{d} \binom{d}{e} = 2^d p^{d/2}(1-p)^{d/2} \end{aligned}$$

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Event error probability

- Hence,

$$\begin{aligned} P_f(E) &< \sum_{d=d_{\text{free}}}^{\infty} A_d[2\sqrt{p(1-p)}]^d \\ &= A(X)|_{X=2\sqrt{p(1-p)}} \end{aligned}$$

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- For small p, the bound is dominated by the first time, thus event error probability can be approximated as

$$P(E) \approx A_{d_{\text{free}}}[2\sqrt{p(1-p)}]^{d_{\text{free}}}$$

Bit error probability

- The bit error probability can be bounded by

$$P_b(E) < \sum_{d=d_{\text{free}}}^{\infty} B_d P_d$$

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- Then we can write

$$P_b(E) < \sum_{d=d_{\text{free}}}^{\infty} B_d[2\sqrt{p(1-p)}]^d = B(X)|_{X=2\sqrt{p(1-p)}}$$

Multiple error events

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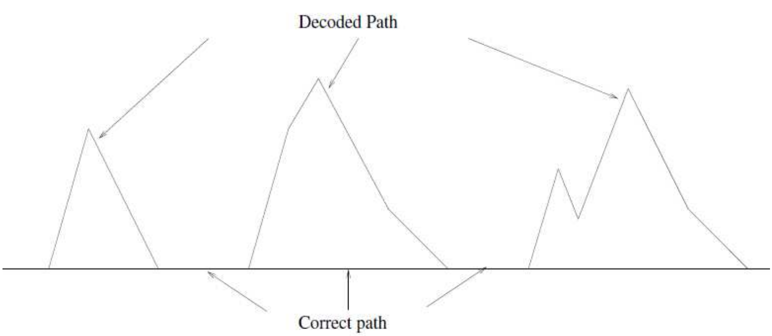
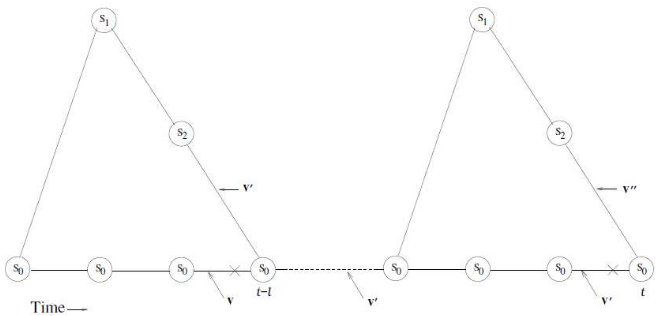


Different error events configuration



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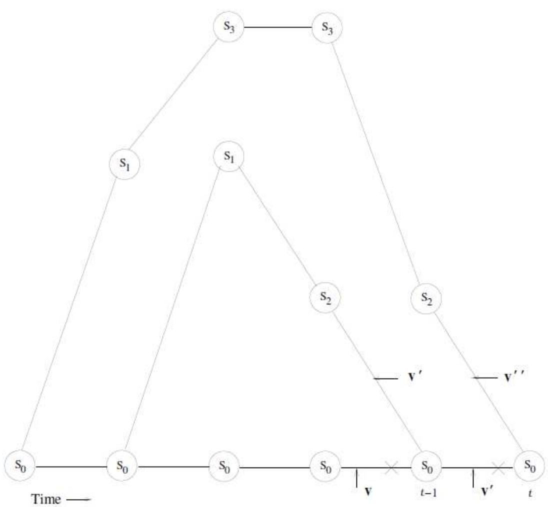

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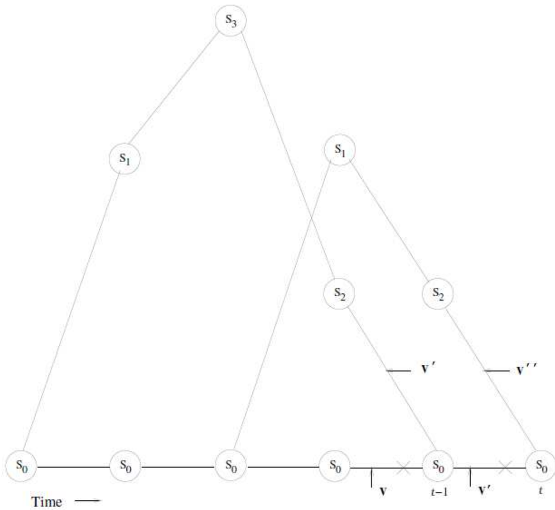

Example: Computation of Event error probability

- For the (3, 1, 2) encoder calculate the event error probability for crossover probability of $p = 10^{-2}$ for binary symmetric channel.

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$$P(E) \approx 2^7 p^{7/2} = 1.28 \times 10^{-5}$$

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B(X) &= (1/k)\frac{\partial A(W,X)}{\partial W}|_{W=1} \\
&= \frac{\partial[X^7 W/(1 - XW - X^3 W)]}{\partial W}|_{W=1} \\
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