An introduction to coding theory

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Lecture #17: Distance Properties of turbo codes

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- Weight distribution of turbo codes

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- Spectral thinning

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- Consider the conventional (2,1,4) convolutional code with generation matrix

$$G_{ff}(D) = \begin{bmatrix} D^4 + D^3 + D^2 + D^1 & D^4 + 1 \end{bmatrix}$$

- This code has $d_{free} = 6$, obtained from the information sequence $u(D) = 1 + D$.
- Now consider the equivalent (2,1,4) systematic recursive convolutional code (SRCC) with generator matrix

$$G_{fb}(D) = \begin{bmatrix} 1 & \dfrac{D^4 + 1}{D^4 + D^3 + D^2 + D + 1} \end{bmatrix}$$

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- The termination bits differ in the two cases.

Introduction

Weight	Multiplicity	Weight	Multiplicity
0	1	17	520
1	0	18	491
2	0	19	346
3	0	20	212
4	0	21	132
5	0	22	68
6	11	23	38
7	12	24	11
8	23	25	2
9	38	26	0
10	61	27	0
11	126	28	0
12	200	29	0
13	332	30	0
14	425	31	0
15	502	32	0
16	545	-	-

Table: Weight spectra of (32,12) terminated convolutional code

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- If a convolutional code has A_d codewords of weight d caused by a set of information sequences $\{u(D)\}$ starting at time 0, it also has A_d codewords of weight d caused by the set of information sequence $\{Du(D)\}$ starting at time 1, A_d codewords of weight d caused by the set $\{D^2u(D)\}$ starting at time 2, and so on.

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- This time invariant property of convolutional encoder accounts for the relatively large number of low weight codewords.

Weight Distribution of Turbo Codes

- Consider Now consider the (2,1,4) systematic recursive convolutional code (SRCC) with generator matrix

$$G_{fb}(D) = \begin{bmatrix} 1 & \dfrac{D^4+1}{D^4+D^3+D^2+D+1} \end{bmatrix}$$

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- An input sequence of length K=16 is fed to the first encoder, producing a parity sequence of length 16.
- The interleaving pattern is given by

$$\pi_{16} = [0, 8, 15, 9, 4, 7, 11, 5, 1, 3, 14, 6, 13, 12, 10, 2]$$
$$(u'_0 = u_0, u'_1 = u_8, u'_2 = u_{15}, \cdots, u'_{15} = u_2)$$

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- Overall rate of this code is $R = 1/3$. To make it a $R = 1/2$ code, we need to puncture every other parity sequence.
- The puncturing matrix is given by

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Weight Distribution of Turbo Codes

Weight	Multiplicity	Weight	Multiplicity
0	1	17	558
1	0	18	478
2	0	19	352
3	0	20	222
4	0	21	123
5	1	22	64
6	4	23	24
7	8	24	4
8	16	25	4
9	30	26	1
10	73	27	0
11	144	28	0
12	210	29	0
13	308	30	0
14	404	31	0
15	496	32	0
16	571	-	-

Table: Weight spectra of (32,12) parallel concatenated code

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- Different interleavers and puncturing matrices would give different weight spectra, but the above result are typical.

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- One can explain the superiority of feedback encoders in parallel concatenation as a consequence of the fact that they are IIR, rather than FIR filters, i.e., their response to single input 1s is not localized to the constraint length of the code but extends over the entire block length.

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- One can explain the superiority of feedback encoders in parallel concatenation as a consequence of the fact that they are IIR, rather than FIR filters, i.e., their response to single input 1s is not localized to the constraint length of the code but extends over the entire block length.
- This property of feedback encoders is exploited by a pseudorandom interleaver to produce the spectral thinning effect.

Distance properties of turbo code

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- Using the concept of a uniform interleaver allows us to calculate the average weight spectrum of a PCC over all possible interleavers.

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- Let us consider a parallel concatenated block code (PCBC) with systematic $(7, 4, 3)$ Hamming code as a constituent code.

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- The input redundancy weight enumerating function (IRWEF) of this systematic code is given by

$$A(W,Z) = W(3Z^2 + Z^3) + W^2(3Z + 3Z^2) + W^3(1 + 3Z) + W^4Z^3.$$

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- The Conditional WEF (CWEF) for each input weight is given by

$$\begin{aligned} A_1(Z) =& 3Z^2 + Z^3, \\ A_2(Z) =& 3Z + 3Z^2, \\ A_3(Z) =& 1 + 3Z, \\ A_4(Z) =& Z^3. \end{aligned}$$

Distance properties of turbo code

- Now consider a PCBC with two systematic (n,k) constituent codes C_1 and C_2 with CWEF's $A_w^{C_1}(Z)$ and $A_w^{C_2}(Z)$ separated by a uniform interleaves of size $N = k$.

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Distance properties of turbo code

- For the $(10, 4)$ PCBC with identical (7,4,3) Hamming constituent codes

$$A_1^{PC}(Z) = \frac{(3Z^2 + Z^3)^2}{\binom{4}{1}} = 2.23Z^4 + 1.5Z^5 + 0.25Z^6$$
$$A_2^{PC}(Z) = \frac{(3Z + 3Z^2)^2}{\binom{4}{2}} = 1.5Z^2 + 3Z^3 + 1.5Z^4$$
$$A_3^{PC}(Z) = \frac{(1 + 3Z)^2}{\binom{4}{3}} = 0.25 + 1.5Z + 2.25Z^2$$
$$A_4^{PC}(Z) = \frac{(Z^3)^2}{\binom{4}{4}} = Z^6$$

Distance properties of turbo code

- Similarly, the IRWEF's are given by

$$\begin{aligned} A^{PC}(W, Z) =& W(2.25Z^4 + 1.5Z^5 + 0.25Z^6) + W^2(1.5Z^2 + 3Z^3 + 1.5Z^4) \\ &+ W^3(0.25 + 1.5Z + 2.25Z^2) + W^4Z^6 \\ B^{PC}(W, Z) =& W(0.56Z^4 + 0.37Z^5 + 0.06Z^6) + W^2(0.75Z^2 + 1.5Z^3 + 0.75Z^4) \\ &+ W^3(0.19 + 1.12 + 1.69Z^2) + W^4Z^6 \end{aligned}$$

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- Also, WEF's are given by

$$\begin{aligned} A^{PC}(X) =& 0.25X^3 + 3X^4 + 7.5X^5 + 3X^6 + 0.25X^7 + X^{10} \\ B^{PC}(X) =& 0.19X^3 + 1.87X^4 + 3.75X^5 + 1.12X^6 + 0.06X^7 + X^{10} \end{aligned}$$

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- The average codeword multiplicities of the low-weight codewords are given by $A_3 = 0.25$ and $A_4 = 3.0$.
- The average bit multiplicities of the low-weight codewords are given by $B_3 = 0.19$ and $B_4 = 1.87$.