An introduction to coding theory

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Lecture #10: Decoding of convolutional codes-I: Viterbi algorithm

Convolutional codes

Outline of the lecture:

- Viterbi decoding of $(n, 1, m)$ code.

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- Viterbi decoding of $(n, 1, m)$ code.
- Example: Viterbi decoding of $(2, 1, 2)$ convolutional code on BSC.

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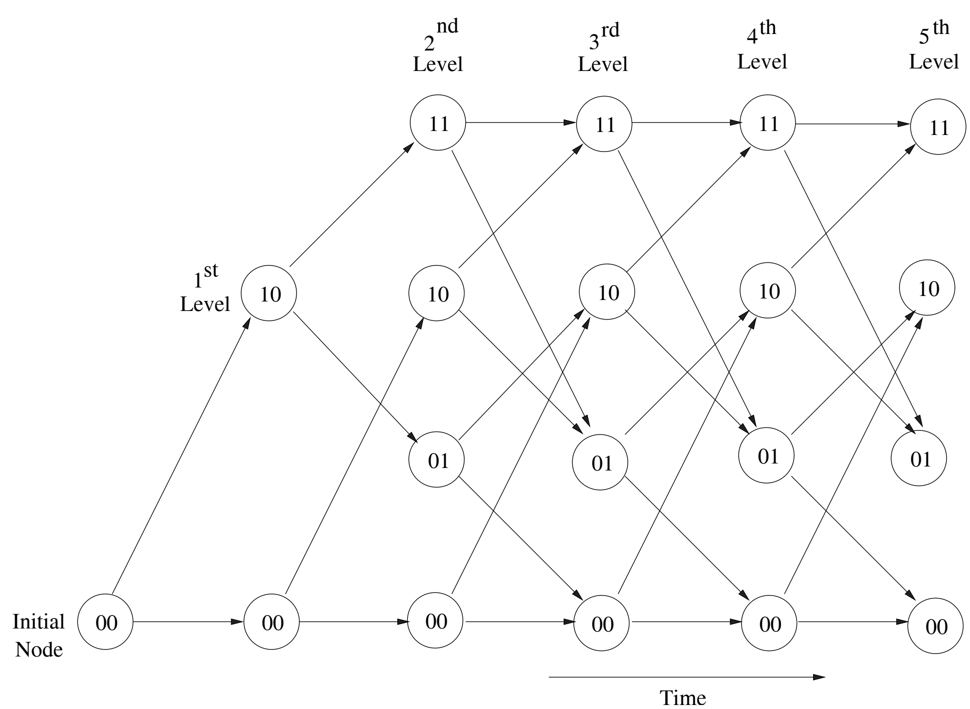
- For time, $l > m$, there are also two branches merging into each state.
- The encoding of a information sequence is equivalent to tracing a path through a trellis.

Viterbi Decoding



Viterbi decoding of $(n, 1, m)$ code

- The encoder is returned to all zero sequence after an L bit information sequence,

$$\mathbf{u} = (u_0, u_1, u_2, \cdots, u_{L-1})$$

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- There are total 2^l paths of length l.

Viterbi decoding of $(n, 1, m)$ code

On BSC:

- Let the information sequence of length L

$$\mathbf{u} = (u_0, u_1, \cdots, u_l, \cdots, u_{L-1})$$

is encoded into code sequence of length $N \triangleq (L+m)n$

$$\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_l, \cdots, \mathbf{v}_{L+m-1})$$

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- If the code sequence $\mathbf{v}$ is transmitted over a channel, let the received sequence is,

$$\mathbf{r} = (\mathbf{r}_0, \mathbf{r}_1, \cdots, \mathbf{r}_l, \cdots, \mathbf{r}_{L+m-1}),$$

where the l^{th} received block is

$$\mathbf{r}_l = (r_l^{(1)}, r_l^{(2)}, \cdots, r_l^{(n)}).$$

Viterbi decoding of $(n, 1, m)$ code

On BSC:

- A maximum likelihood decoder finds a path through the trellis that maximizes the path conditional probability

$$P(\mathbf{r}|\mathbf{v}) = \prod_{l=0}^{L+m-1} P(\mathbf{r}_l|\mathbf{v}_l)$$

where the branch conditional probability

$$P(\mathbf{r}_l|\mathbf{v}_l) = \prod_{l=1}^{n} P(r_l^{(i)}|v_l^{(i)})$$

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- The bit conditional probabilities $P(r_l^{(i)}|v_l^{(i)})$ are the channel transition probabilities.

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$$M(\mathbf{r}|\mathbf{v}) \triangleq \log P(\mathbf{r}|\mathbf{v})$$

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- Maximizing $P(\mathbf{r}|\mathbf{v})$ is equivalent to maximizing

$$M(\mathbf{r}|\mathbf{v}) \triangleq \log P(\mathbf{r}|\mathbf{v})$$

- $M(\mathbf{r}|\mathbf{v})$ is called the path metric.

$$\begin{aligned} M(\mathbf{r}|\mathbf{v}) &= \sum_{l=0}^{L+m-1} \log P(\mathbf{r}_l|\mathbf{v}_l) \\ &= \sum_{l=0}^{L+m-1} M(\mathbf{r}_l|\mathbf{v}_l), \qquad \text{(branch metrics)} \end{aligned}$$

$$\begin{aligned} M(\mathbf{r}_l|\mathbf{v}_l) &= \sum_{i=1}^{n} \log P(r_l^{(i)}|v_l^{(i)}) \\ &= \sum_{i=1}^{n} M(r_l|v_l), \qquad \text{(bit metrics)} \end{aligned}$$

Viterbi decoding of $(n, 1, m)$ code

- The partial path metric for the first j branches of a path $\mathbf{v}$ is given by

$$M([\mathbf{r}|\mathbf{v}]_j) = \sum_{l=0}^{j-1} M(\mathbf{r}_l|\mathbf{v}_l)$$

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- For BSC, the maximum likelihood decoder decodes the received sequence $\mathbf{r}$ into code sequence $\mathbf{v}$ that minimizes the Hamming distance $d(\mathbf{r}, \mathbf{v})$

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- For BSC, the maximum likelihood decoder decodes the received sequence $\mathbf{r}$ into code sequence $\mathbf{v}$ that minimizes the Hamming distance $d(\mathbf{r}, \mathbf{v})$
- The Viterbi algorithm is a computationally efficient method of finding the path through the trellis with the best metric.

Viterbi decoding of $(n, 1, m)$ code

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- The Viterbi decoder proceeds through the trellis level by level in search of the path with the best metric.
- At each level, the decoder compares the metric of all partial paths entering each state.
- The decoder stores the partial path entering each state with the best metric (survivor path) and eliminates all other partial paths.

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- For $m \leq l \leq L$, there are total 2^m survivors.

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- The number of survivors decrease during the termination process, until at time $l = L + m$ when there is only one survivor left.

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- The number of survivors decrease during the termination process, until at time $l = L + m$ when there is only one survivor left.
- the surviving path is the maximum likelihood path.

Viterbi decoding of $(n, 1, m)$ code

Viterbi Algorithm:

- Step 1: Starting at level $l = m$ in the trellis, compute the partial metric for the single path entering each m^{th} level state. Store the survivor path and its metric for each state.

Viterbi decoding of $(n, 1, m)$ code

Viterbi Algorithm:

- Step 1: Starting at level $l = m$ in the trellis, compute the partial metric for the single path entering each m^{th} level state. Store the survivor path and its metric for each state.
- Step 2: Increase time l by one. Compute the partial metric for all the paths entering at the $(l + 1)^{th}$ level state by adding the branch metric entering that state to the metric of the connecting survivor path at the previous l^{th} level state. Store the survivor path and its metric for each state.

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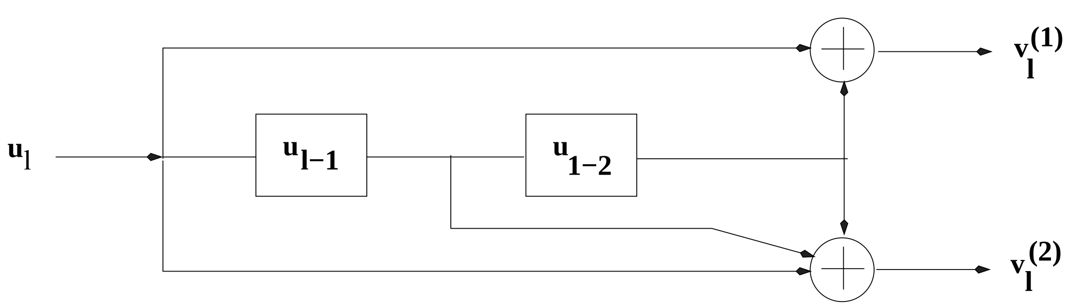
- Step 1: Starting at level $l = m$ in the trellis, compute the partial metric for the single path entering each m^{th} level state. Store the survivor path and its metric for each state.
- Step 2: Increase time l by one. Compute the partial metric for all the paths entering at the $(l+1)^{th}$ level state by adding the branch metric entering that state to the metric of the connecting survivor path at the previous l^{th} level state. Store the survivor path and its metric for each state.
- Step 3: Repeat Step 2 until you are at the end of the trellis $(l = L + m)$.

Viterbi Decoding

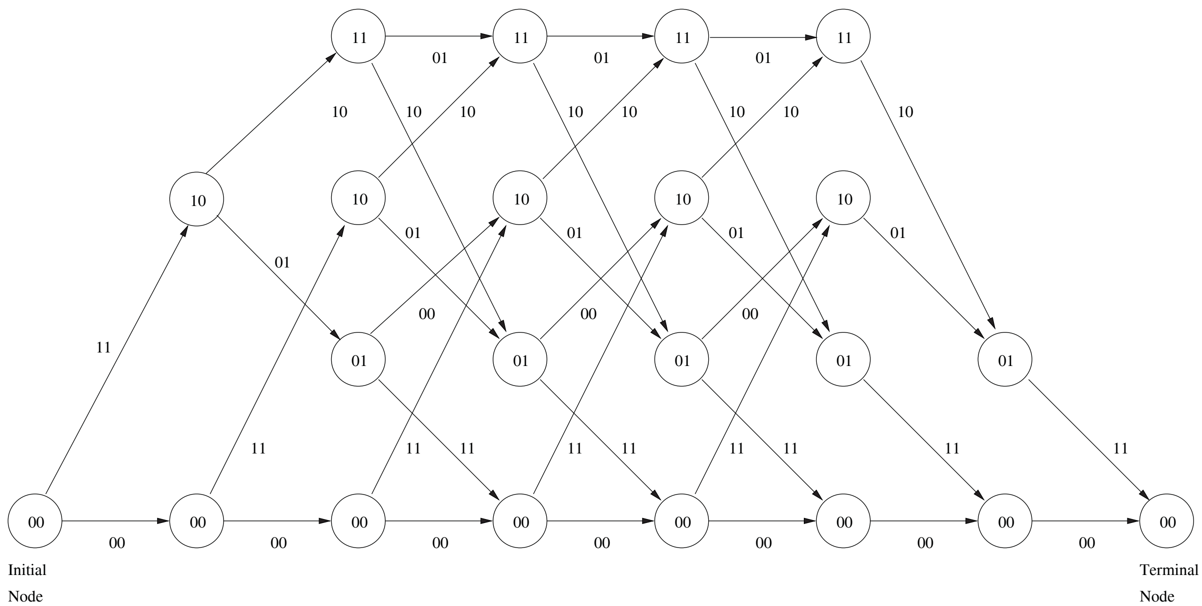
Example:



- This $(2, 1, 2)$ convolutional code with $L = 5$ is used on a BSC. The received sequence is

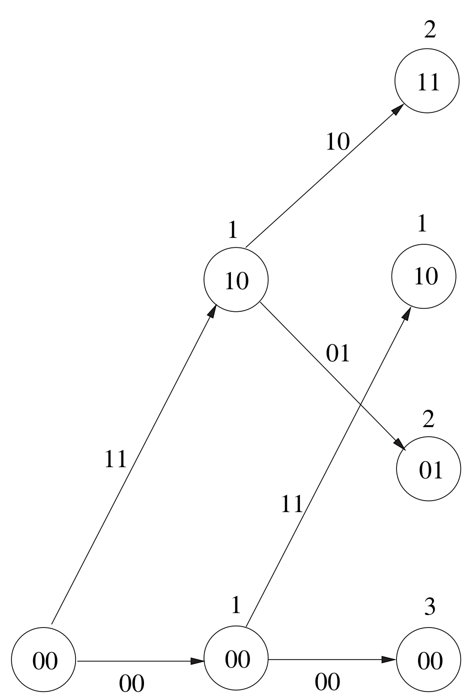
$$\mathbf{r} = (01,\ 11,\ 10,\ 10,\ 00,\ 11,\ 10)$$

Viterbi Decoding



Trellis diagram of $(2, 1, 2)$ convolutional code with $L = 5$.

Viterbi Decoding



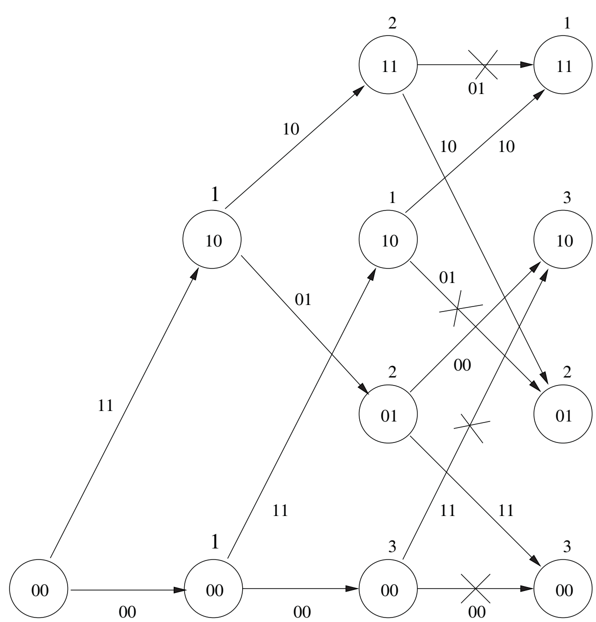
Level 2

Viterbi Decoding

$$\mathbf{r} = (01,\ 11\ )$$

$$\mathbf{v}_1 = (00,\ 00\ ) \qquad d(\mathbf{v}_1, \mathbf{r}) = 3$$
$$\mathbf{v}_2 = (00,\ 11\ ) \qquad d(\mathbf{v}_2, \mathbf{r}) = 1$$
$$\mathbf{v}_3 = (11,\ 01\ ) \qquad d(\mathbf{v}_3, \mathbf{r}) = 2$$
$$\mathbf{v}_4 = (11,\ 10\ ) \qquad d(\mathbf{v}_4, \mathbf{r}) = 2$$

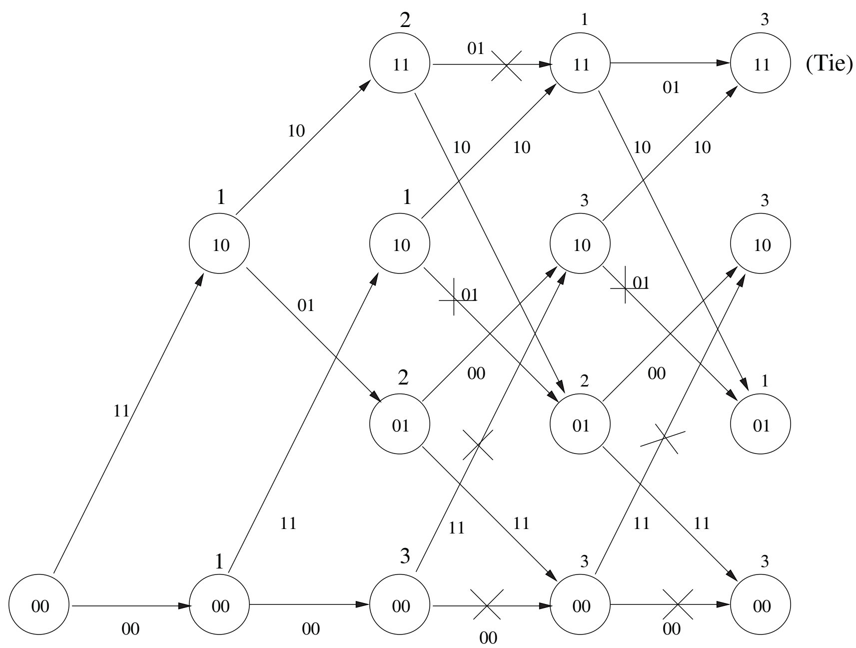
Viterbi Decoding



$$\mathbf{r} = (01,\ \ 11\ \ 10)$$

Level 3

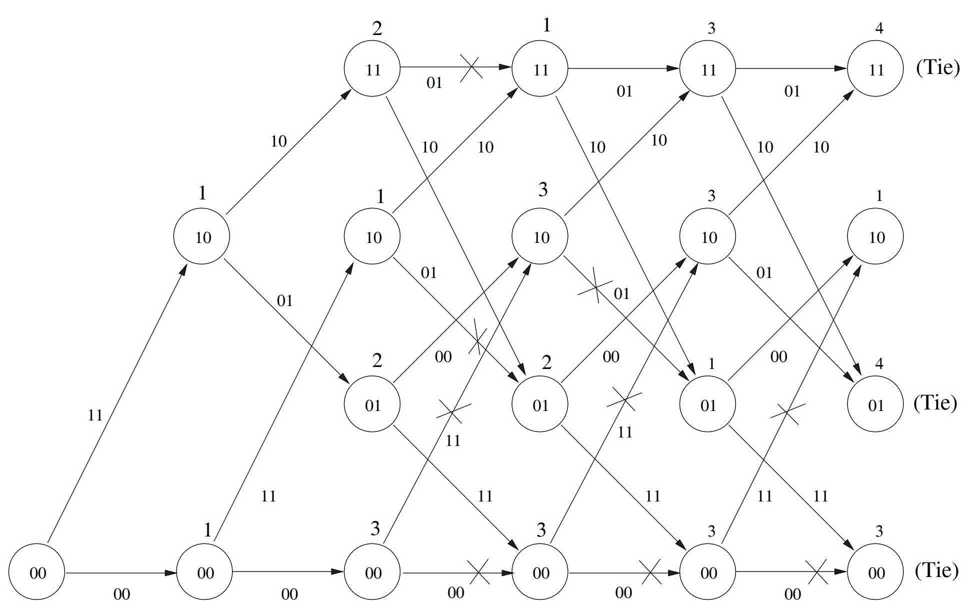
Viterbi Decoding



$\mathbf{r} = (01,\ \ 11\ \ 10\ \ 10)$

Level 4

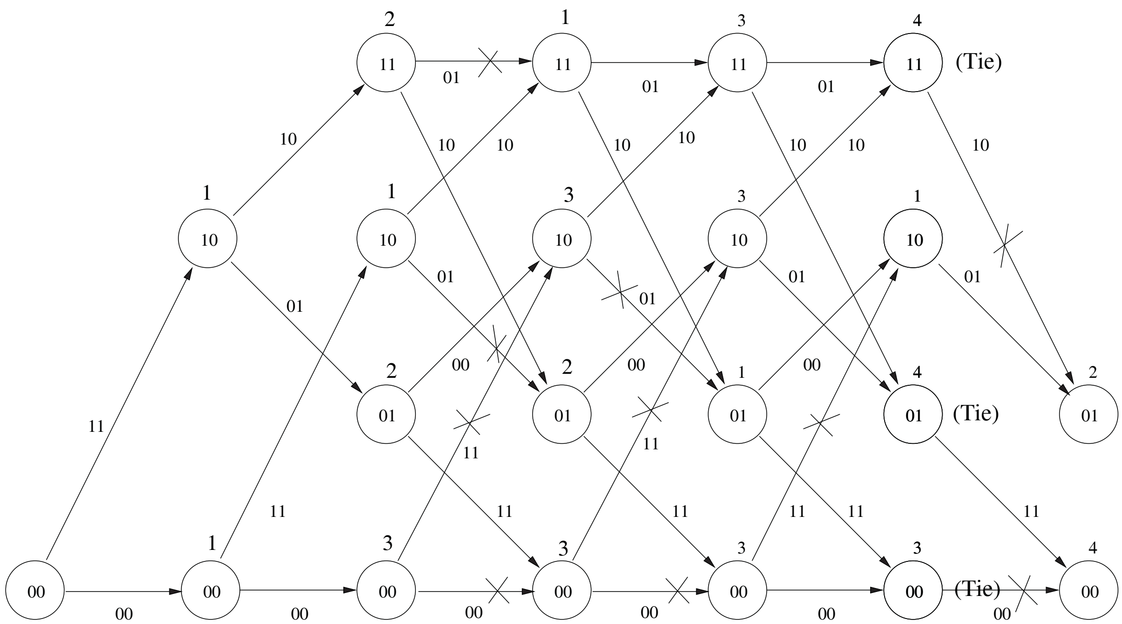
Viterbi Decoding



$\mathbf{r} = (01,\ \ 11\ \ 10\ \ 10\ \ 00)$

Level 5

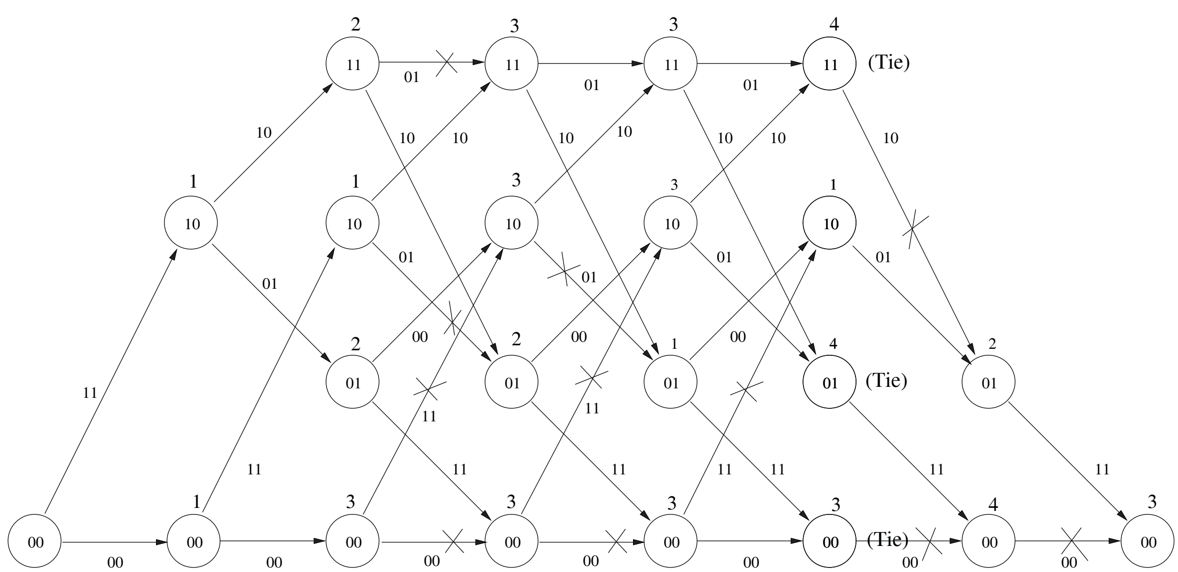
Viterbi Decoding



$\mathbf{r} = (01,\ \ 11\ \ 10\ \ 10\ \ 00\ \ 11)$

Level 6

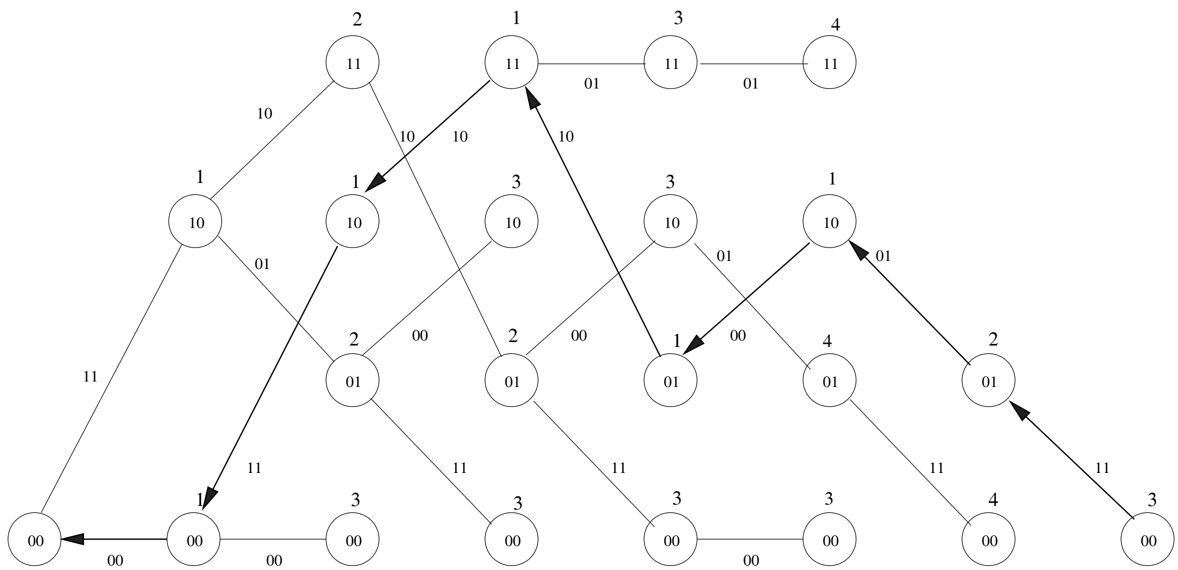
Viterbi Decoding



$\mathbf{r} = (01,\ \ 11\ \ 10\ \ 10\ \ 00\ \ 11\ \ 10)$

Level 7

Viterbi Decoding



$$\hat{\mathbf{v}} = (00,\ \ 11\ \ 10\ \ 10\ \ 00\ \ 01\ \ 11)$$

Level 7