An introduction to coding theory

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Lecture #11B: Problem solving session-IV

Convolutional codes

- **Problem # 1:** Consider rate $R = 1/3$ convolutional code with encoding matrix

$$G(D) = \begin{pmatrix} \frac{1+D+D^2}{1+D^4} & \frac{1+D^2+D^4}{1+D^4} & \frac{1+D+D^2}{1+D^2} \end{pmatrix}$$

Convolutional codes

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a) Is G(D) catastrophic? Explain.

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  a) Is G(D) catastrophic? Explain.

- **Solution:** Yes, G(D) can be equivalently written as

$$G(D) = \frac{1}{1+D^2}\left[\frac{1+D+D^2}{1+D^2} \quad \frac{1+D^2+D^4}{1+D^2} \quad 1+D+D^2\right]$$

or equivalently in the form

$$G(D) = \frac{1+D+D^2}{1+D^4}\left[1 \quad 1+D+D^2 \quad 1+D^2\right]$$

An input of infinite weight $\frac{1+D^4}{1+D+D^2}$ can result in an output sequence of weight 6.

Convolutional codes

b) Find a minimal encoder whose encoding matrix is equivalent to G(D).
Solution: An equivalent minimal encoder is given by

$$G(D) = \left[1 \quad 1+D+D^2 \quad 1+D^2\right]$$

Convolutional codes

- **Problem # 2:**

Convolutional codes

- **Problem # 2:**
  a) Consider the rate R=2/3 nonsystematic feedforward encoder with generator matrix given by

$$G(D) = \begin{bmatrix} D & D^2 & 1 \\ 1 & D^2 & 1+D+D^2 \end{bmatrix}$$

Convolutional codes

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  i) Draw the controller canonical form encoder realization for G(D).

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- **Solutions:**

- **Problem # 2:**
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  i) Draw the controller canonical form encoder realization for G(D).
- **Solutions:**
  i) Controller canonical form encoder realization is given in Figure 1

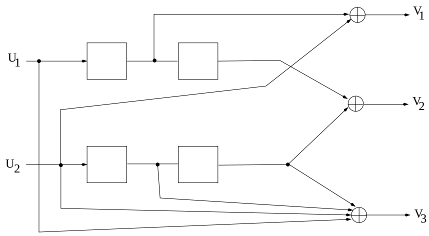

Figure: Canonical Form encoder realization

ii) Find the generator matrix G'(D) of the equivalent systematic feedback encoder.

ii) **Solutions:** We have

$$G(D) = \begin{bmatrix} D & D^2 & 1 \\ 1 & D^2 & 1+D+D^2 \end{bmatrix}$$

Convolutional codes

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ii) **Solutions:** We have

$$G(D) = \begin{bmatrix} D & D^2 & 1 \\ 1 & D^2 & 1+D+D^2 \end{bmatrix}$$

- Now applying Row 1 $\leftarrow$ (Row 1)/D, we get

$$G(D) = \begin{bmatrix} 1 & D & \frac{1}{D} \\ 1 & D^2 & 1+D+D^2 \end{bmatrix}$$

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- Now applying Row 2 $\leftarrow$ Row 1 + Row 2, we get

$$G(D) = \begin{bmatrix} 1 & D & \frac{1}{D} \\ 0 & D+D^2 & \frac{1+D+D^2+D^3}{D} \end{bmatrix}$$

- Now applying Row 2 $\leftarrow$ (Row 2)/$(D+D^2)$, we get

$$G(D) = \begin{bmatrix} 1 & D & \frac{1}{D} \\ 0 & 1 & \frac{1+D^2}{D^2} \end{bmatrix}$$

- Now applying Row 1 $\leftarrow$ Row 1 + D (Row 2), we get

$$G(D) = \begin{bmatrix} 1 & 0 & D \\ 0 & 1 & \frac{1+D^2}{D^2} \end{bmatrix}$$

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iii) Is G'(D) realizable? If not, find an equivalent realizable generator matrix and draw the corresponding minimal encoder realization.

iii) Equivalent systematic encoder is not realizable, so we can make it realizable by applying Row 2 $\leftarrow$ (Row 2)(D^2). We get

$$G(D) = \begin{bmatrix} 1 & 0 & D \\ 0 & D^2 & 1+D^2 \end{bmatrix}$$

Convolutional codes

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iii) Equivalent systematic encoder is not realizable, so we can make it realizable by applying Row 2 $\leftarrow$ (Row 2)(D^2). We get

$$G(D) = \begin{bmatrix} 1 & 0 & D \\ 0 & D^2 & 1+D^2 \end{bmatrix}$$

- Controller canonical form encoder realization of the equivalent encoder is given in Figure 2

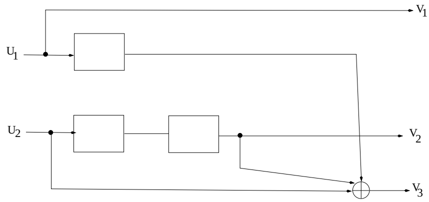


Figure: Canonical Form encoder realization of the equivalent encoder

Convolutional codes

- **Problem # 3:** In Figure 3, a rate $R = 2/3$ systematic convolutional encoder is shown.

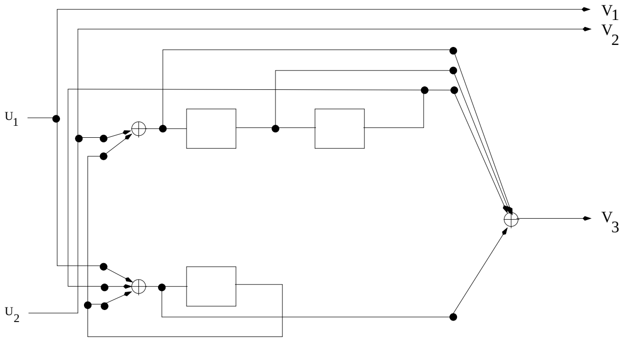


Figure: Figure for Problem 3.

Convolutional codes

(a) Give the expression for the generator matrix $\mathbf{G}(\mathbf{D})$

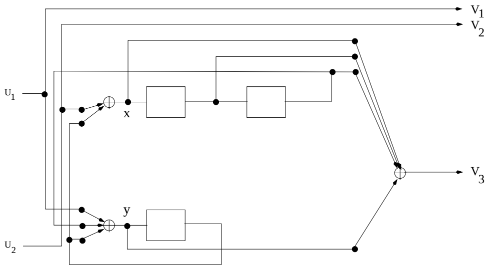
- **Solutions:**

Convolutional codes

(a) Give the expression for the generator matrix $\mathbf{G}(\mathbf{D})$

- **Solutions:**
- We can write

$$
\begin{aligned}
v_1(D) &= u_1(D) \\
v_2(D) &= u_2(D) \\
v_3(D) &= (1 + D + D^2)x(D) + y(D)
\end{aligned}
$$



Convolutional codes

- Also,

$$
\begin{aligned}
y(D) &= u_1(D) + D^2x(D) + Dy(D) \\
x(D) &= u_2(D) + Dy(D)
\end{aligned}
$$

- Solving for x(D) and y(D), we get

$$
\begin{aligned}
y(D) &= \frac{1}{1 + D + D^3}u_1(D) + \frac{D^2}{1 + D + D^3}u_2(D) \\
x(D) &= \frac{D}{1 + D + D^3}u_1(D) + \frac{1 + D}{1 + D + D^3}u_2(D)
\end{aligned}
$$

- Hence, $v_3(D)$ is

$$
y(D) = \frac{1 + D + D^2 + D^3}{1 + D + D^3}u_1(D) + \frac{1 + D^2 + D^3}{1 + D + D^3}u_2(D)
$$

- Therefore, $\mathbf{G(D)}$ is given by

$$\mathbf{G(D)} = \begin{bmatrix} 1 & 0 & \frac{1+D+D^2+D^3}{1+D+D^3} \\ 0 & 1 & \frac{1+D^2+D^3}{1+D+D^3} \end{bmatrix}$$

(b) Realize the convolution encoder in controller canonical form. How many termination bits are required to terminate the encoder for controller canonical form realization.

- Convolutional encoder in controller canonical form.

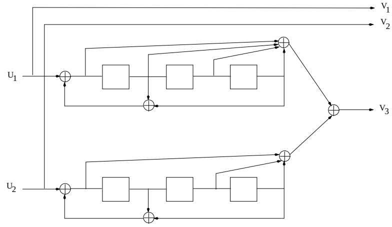


Figure: Answer to Problem 3(b)

Convolutional codes

(b) Realize the convolution encoder in controller canonical form. How many termination bits are required to terminate the encoder for controller canonical form realization.

- Convolutional encoder in controller canonical form.

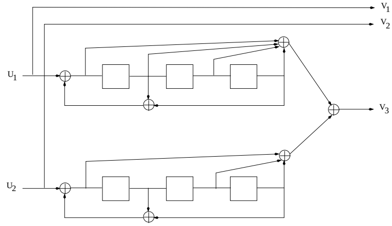


Figure: Answer to Problem 3(b)

- The controller canonical form realization of this encoder uses 6 memory elements, so the encoder will require 6 termination bits to return to all-zero state.

Convolutional codes

- **Problem # 4:**

Convolutional codes

- **Problem # 4:**
  (a) Show that the a-priori probability can be written in this form

$$P(u_l = \pm 1) = A_l \exp^{u_l L_a(u_l)/2}$$

  where $L_a(\cdot)$ is the a-priori L-values of the information bits.

Convolutional codes

- **Problem # 4:**
  (a) Show that the a-priori probability can be written in this form

$$P(u_l = \pm 1) = A_l \exp^{u_l L_a(u_l)/2}$$

  where $L_a(\cdot)$ is the a-priori L-values of the information bits.
  (b) Using this result, show that the branch metrics $\gamma^*(s, s')$ for a continuous-output AWGN channel (in log-domain) can be written as

$$\gamma^*(s, s') = \frac{u_l L_a(u_l)}{2} + L_c \mathbf{r}_l \cdot \mathbf{v}_l$$

  where $L_c = 4E_s/N_0$ is the channel reliability factor. Notations are the same as used in class lectures.

Convolutional codes

- **Solutions:** We can write

$$
\begin{aligned}
P(u_l = \pm 1) &= \frac{[P(u_l = +1)/P(u_l = -1)]^{\pm 1}}{\{1 + [P(u_l = +1)/P(u_l = -1)]^{\pm 1}\}} \\
&= \frac{e^{\pm L_a(u_l)}}{\{1 + e^{\pm L_a(u_l)}\}} \\
&= \frac{e^{-L_a(u_l)/2}}{\{1 + e^{-L_a(u_l)}\}} e^{u_l L_a(u_l)/2} \\
&= A_l e^{u_l L_a(u_l)/2},
\end{aligned}
$$

Convolutional codes

- Also,

$$
\begin{aligned}
\gamma_l(s', s) &= A_l e^{u_l L_a(u_l)/2} e^{-(E_s/N_0)||\mathbf{r}_l - \mathbf{v}_l||^2}, \\
&= A_l e^{u_l L_a(u_l)/2} e^{(2E_s/N_0)(\mathbf{r}_l \cdot \mathbf{v}_l) - (E_s/N_0)(||\mathbf{r}_l||^2 + ||\mathbf{v}_l||^2)} \\
&= A_l e^{-(E_s/N_0)(||\mathbf{r}_l||^2 + ||\mathbf{v}_l||^2)} e^{u_l L_a(u_l)/2} e^{(L_c/2)(\mathbf{r}_l \cdot \mathbf{v}_l)} \\
&= A_l B_l e^{u_l L_a(u_l)/2} e^{(L_c/2)(\mathbf{r}_l \cdot \mathbf{v}_l)}
\end{aligned}
$$

Convolutional codes

- Also,

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&= A_l e^{-(E_s/N_0)(||\mathbf{r}_l||^2 + ||\mathbf{v}_l||^2)} e^{u_l L_a(u_l)/2} e^{(L_c/2)(\mathbf{r}_l \cdot \mathbf{v}_l)} \\
&= A_l B_l e^{u_l L_a(u_l)/2} e^{(L_c/2)(\mathbf{r}_l \cdot \mathbf{v}_l)}
\end{aligned}
$$

- Thus

$$
\gamma_l^*(s', s) = \ln \gamma_l(s', s) = \frac{u_l L_a(u_l)}{2} + \frac{L_c}{2} \mathbf{r}_l \cdot \mathbf{v}_l
$$