An introduction to coding theory

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Lecture #9A: Convolutional codes: Classification, Realization

Outline of the lecture

- Classification of convolutional encoder

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  - Feedforward encoder

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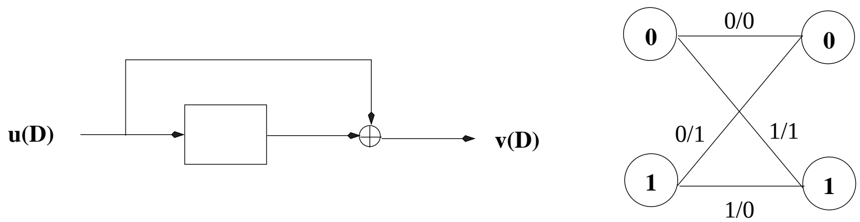
- Minimal encoder

Classification of convolutional encoders

Feedforward Encoder:

- The encoder corresponding to a polynomial generator matrix does not contain any feedback path, and hence it is known as a feedforward encoder.



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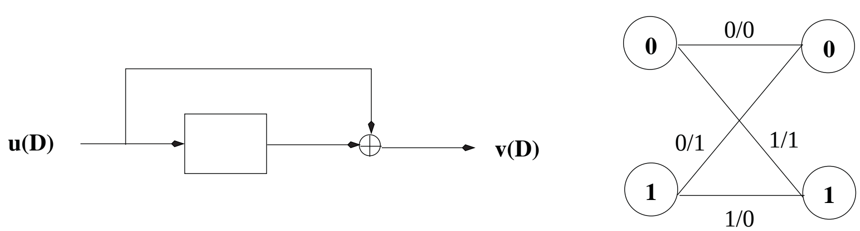

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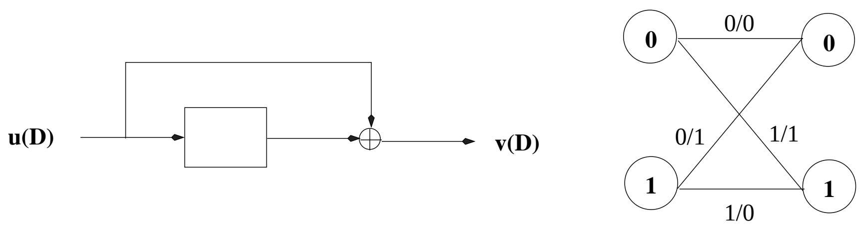
- In figure, the encoder diagram of a rate $R = 1$, 2-state feedforward encoder with generator matrix $\mathbf{G}(D) = [1 + D]$ is shown using a shift register implementation.



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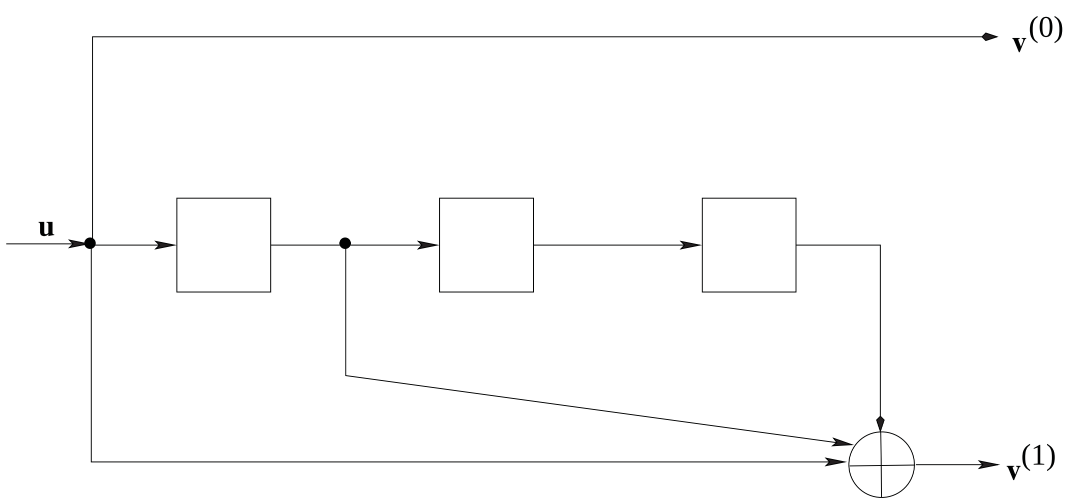
Classification of convolutional encoders

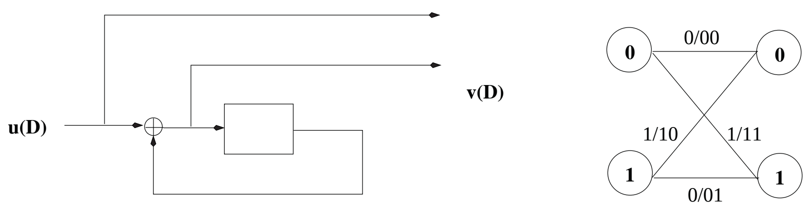


Example: Feedforward encoder

Classification of convolutional encoders

Feedback Encoder:

- The encoder corresponding to a rational generator matrix with atleast one nonpolynomial transfer function contains a feedback path and is known as a feedback encoder.



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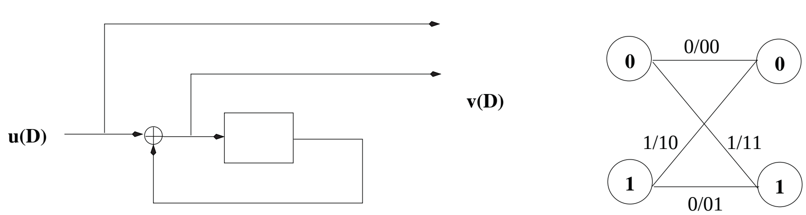

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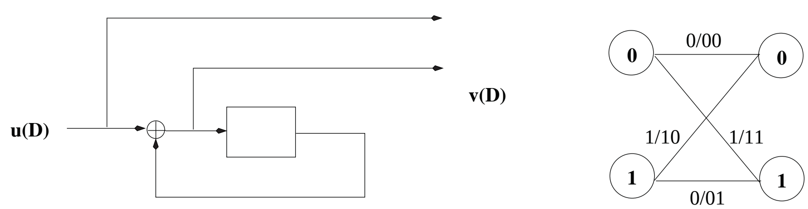

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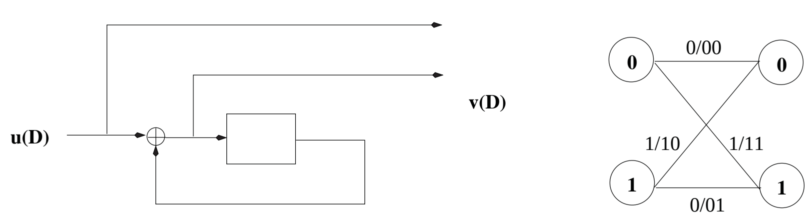
- In figure, the encoder diagram of a rate $R = 1/2$, 2-state feedback encoder with generator matrix $\mathbf{G}(D) = [1 \ \ \frac{1}{1+D}]$ is shown.



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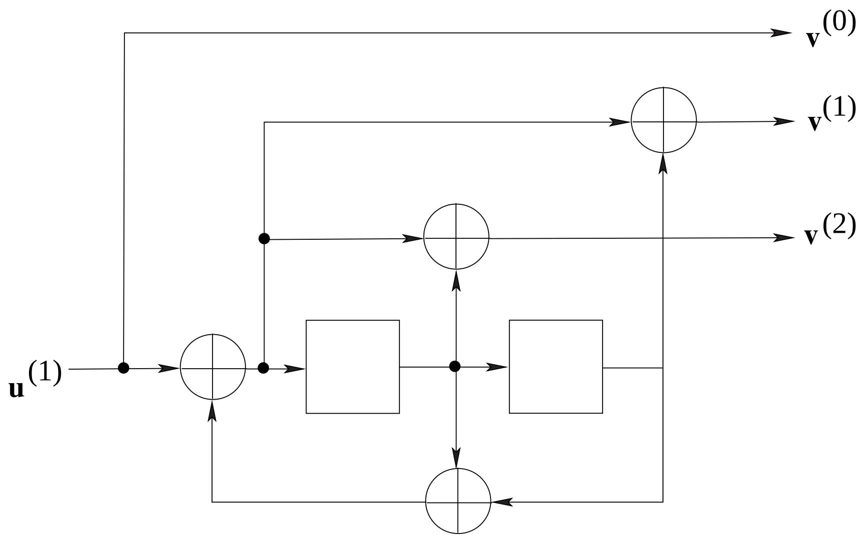
Classification of convolutional encoders



(c)

Example: Feedback encoder

Classification of convolutional encoders

Systematic Encoder:

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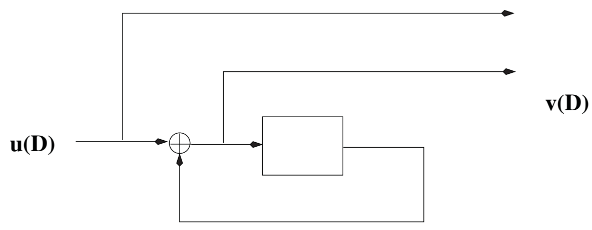
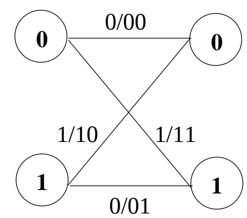


(a)



(b)

Classification of convolutional encoders

Nonsystematic Encoder:

- In a nonsystematic convolutional encoder, the k information sequences do not appear unchanged in the n code sequences.

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Classification of convolutional encoders

Nonsystematic Encoder:

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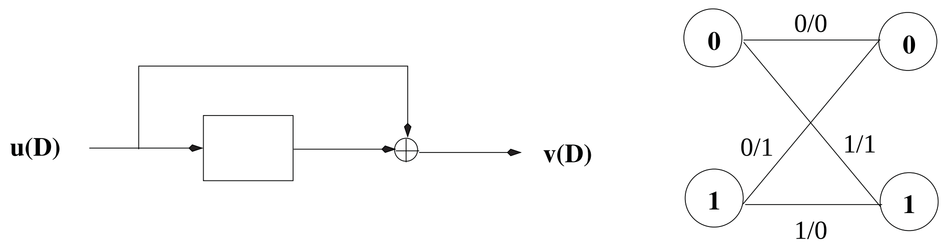
- In figure, a nonsystematic rate $R = 1/2$ feedforward convolutional encoder is shown.



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Equivalent Encoder

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- Example: The generator matrix, $\mathbf{G}(D) = [1 \quad \frac{1}{1+D}]$ and $\mathbf{G}^{'}(D) = [1 + D \quad 1]$ are equivalent.

Equivalent Encoder

- Consider the nonsystematic encoder

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- Step 1: Row 1 $\implies$ [1/(1+D)][Row 1].

$$\mathbf{G}_1(D) = \begin{bmatrix} 1 & D/(1+D) & 1 \\ D & 1 & 1 \end{bmatrix}$$

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- Step 2: Row 2 $\implies$ Row 2 + [D][Row 1].

$$\mathbf{G}_2(D) = \begin{bmatrix} 1 & D/(1+D) & 1 \\ 0 & (1+D+D^2)/(1+D) & 1+D \end{bmatrix}$$

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- Step 3: Row 2 $\implies$ [(1+D)/$(1+D+D^2)$][Row 2].

$$\mathbf{G}_3(D) = \begin{bmatrix} 1 & D/(1+D) & 1 \\ 0 & 1 & (1+D^2)/(1+D+D^2) \end{bmatrix}$$

Equivalent Encoder

- Step 4: Row 1 $\implies$ Row 1 + [D/(1+D)][Row 2].

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Equivalent Encoder

- Step 4: Row 1 $\Longrightarrow$ Row 1 + [D/(1+D)][Row 2].
- Modified Systematic generator matrix

$$\mathbf{G}'(D) = \left[\begin{array}{ccc} 1 & 0 & 1/(1+D+D^2) \\ 0 & 1 & (1+D^2)/(1+D+D^2) \end{array} \right]$$

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- Example:

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- If the input sequence $\mathbf{u}(D) = \left[\frac{1}{1+D}\right] = 1 + D + D^2 + \cdots$, then the output sequence, $[1 \quad 1+D]$ has only weight 3, even thought the information sequence has infinite weight.

Controller Canonical Form Realization

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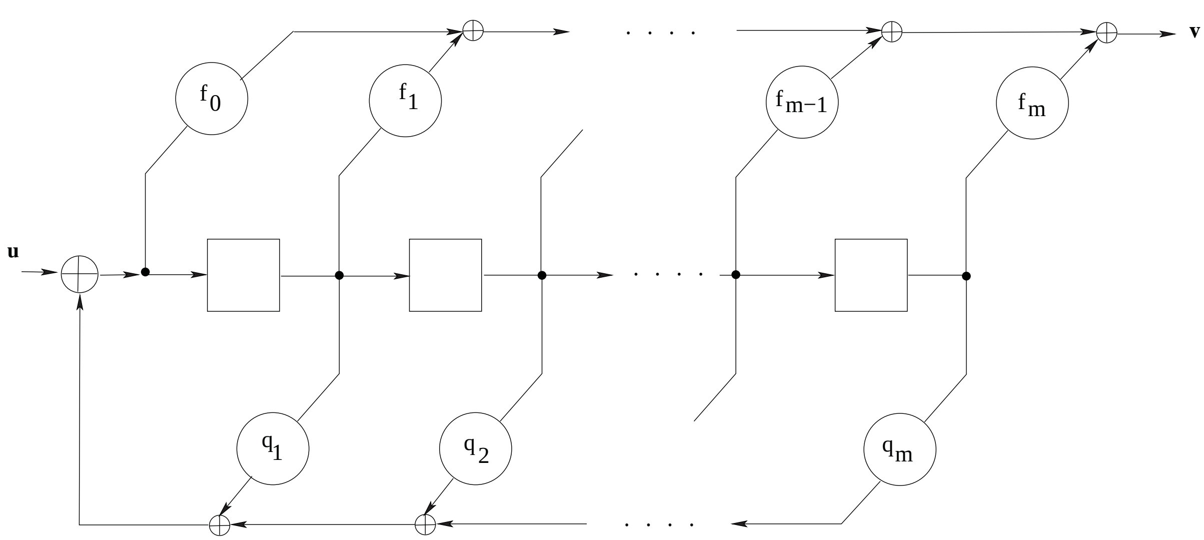
- The n adders used to obtain output sequences are external to the shift register.
- In Figure 3.6(a) (next page), a rate $R = 1$, nonsystematic convolutional encoder with following generator function $\mathbf{G}(D)$ is implemented in controller canonical form realization.

$$\mathbf{G}(D) = \left[\frac{f_0 + f_1 D + \cdots + f_{m-1} D^{m-1} + f_m D^m}{1 + q_1 D + q_2 D^2 + \cdots + q_m D^m} \right]$$

Controller Canonical Form Realization



(a)

Figure 3.6

Observer Canonical Form Realization

- In observer canonical form realization, to realize a rate $R = k/n$ convolutional encoder, n shift register are used for output sequences.

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- In observer canonical form realization, to realize a rate $R = k/n$ convolutional encoder, n shift register are used for output sequences.
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- The lowest degree term in the generator polynomial represent the connections to the right hand side of the shift registers.

Observer Canonical Form Realization

- In observer canonical form realization, to realize a rate $R = k/n$ convolutional encoder, n shift register are used for output sequences.
- The k input sequences enter the adders internal to the shift registers.
- The lowest degree term in the generator polynomial represent the connections to the right hand side of the shift registers.
- In Figure 3.6(b) (next page), a rate $R = 1$, nonsystematic convolutional encoder with following generator function $\mathbf{G}(D)$ is implemented in observer canonical form realization.

$$\mathbf{G}(D) = \left[\frac{f_0 + f_1 D + \cdots + f_{m-1} D^{m-1} + f_m D^m}{1 + q_1 D + q_2 D^2 + \cdots + q_m D^m} \right]$$

Observer Canonical Form Realization

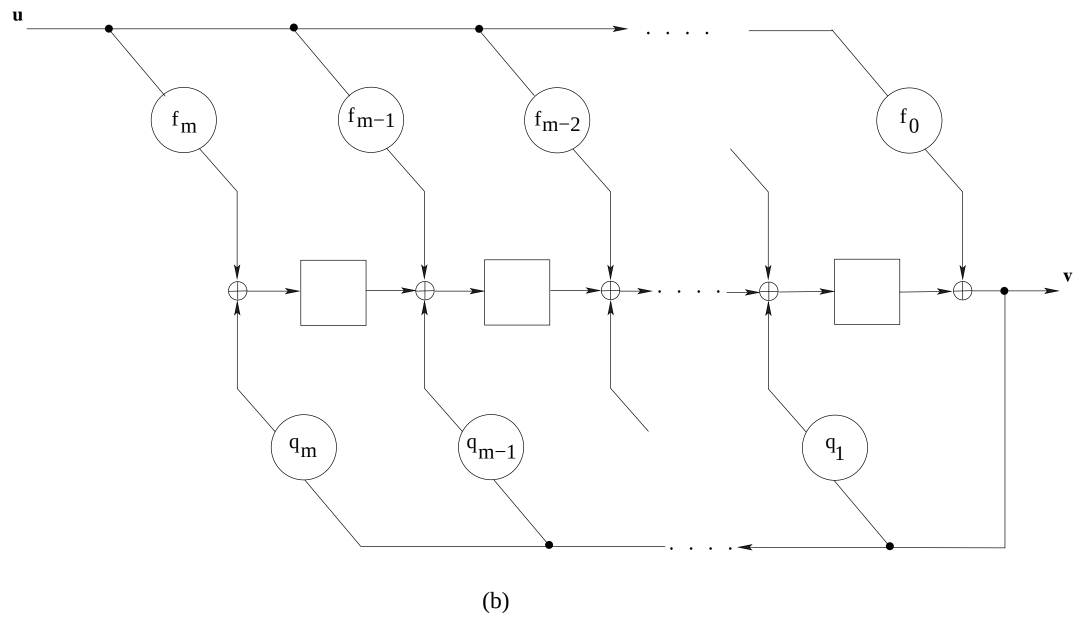

Figure 3.6

Realization of Convolutional encoder

Example 3.7:

- Let's consider a rate $R = 2/3$ systematic feedforward encoder with generator matrix

$$\mathbf{G}(D) = \begin{bmatrix} 1 & 0 & 1+D+D^2 \\ 0 & 1 & 1+D \end{bmatrix}.$$

Realization of Convolutional encoder

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$$\mathbf{G}(D) = \begin{bmatrix} 1 & 0 & 1+D+D^2 \\ 0 & 1 & 1+D \end{bmatrix}.$$

- The parity check matrix can be written as

$$\mathbf{H}(D) = \begin{bmatrix} \mathbf{h}^{(0)}(D) & \mathbf{h}^{(1)}(D) & 1 \end{bmatrix} = \begin{bmatrix} 1+D+D^2 & 1+D & 1 \end{bmatrix}.$$

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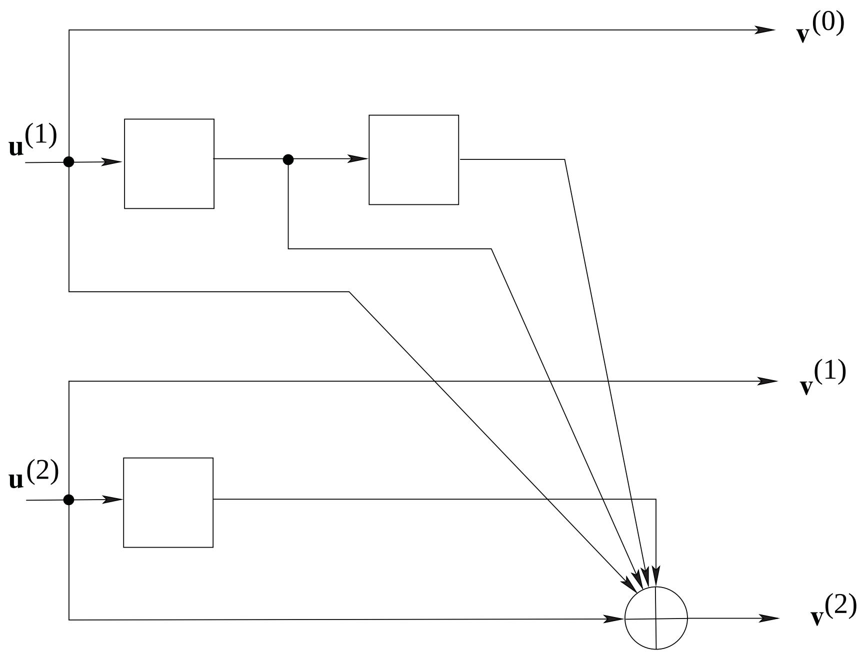
- The controller canonical form realization results in $(3, 2, 3)$ encoder.
- The observer canonical form realization results in $(3, 2, 2)$ encoder.

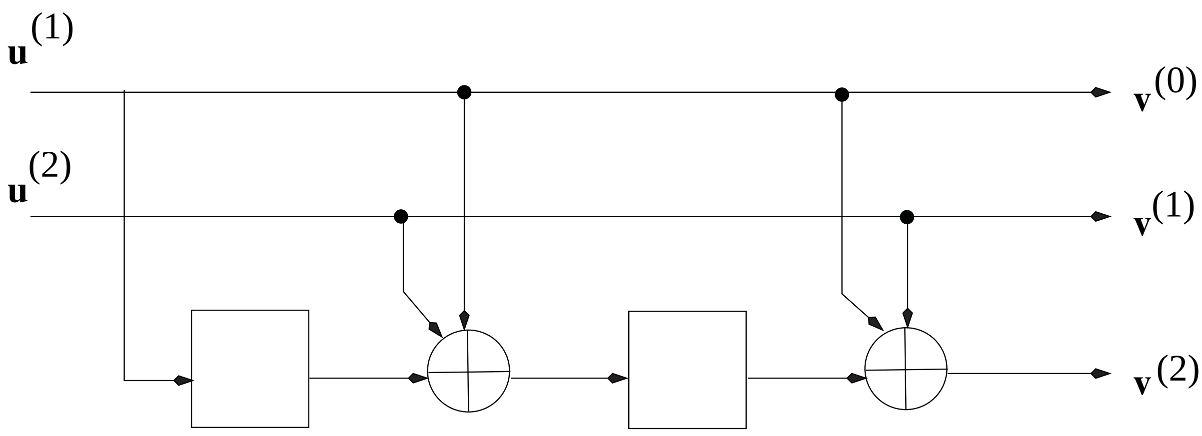
Controller Canonical Form Realization



(a)

Example 3.7

Observer Canonical Form Realization



(b)

Example 3.7

Minimal encoder

- A generator matrix of a convolutional code is minimal if its number of states is minimal over all equivalent generator matrices.

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- A generator matrix of a convolutional code is minimal if its number of states is minimal over all equivalent generator matrices.
- A minimal encoder is a realization of a minimal encoding matrix $\mathbf{G}(D)$ with the minimal number of memory elements over all realizations of $\mathbf{G}(D)$.