

## Week 2. Lecture Notes

Topics: The Master Method  
Divide and Conquer  
Strassen's Algorithm  
Quick Sort

### The Master Method

The Master Method applies to recurrences of the form

$$T(n) = aT(n/b) + f(n)$$

where  $a \geq 1$ ,  $b > 1$  and  $f$  is asymptotically positive

### Three Common Cases

Compare  $f(n)$  with  $n^{\log_b a}$

1.  $f(n) = O(n^{\log_b a - \epsilon})$  for some constant  $\epsilon > 0$ ,

$f(n)$  grows polynomially slower than  ~~$n^{\log_b a}$~~   
 $n^{\log_b a}$  by a  $n^\epsilon$  factor

Solution:  $T(n) = \Theta(n^{\log_b a})$

2.  $f(n) = \Theta(n^{\log_b a} \lg^k n)$  for some constant  $k \geq 0$   
 $f(n)$  and  $n^{\log_b a}$  grows at similar rates

**Solution:**  $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

3.  $f(n) = \Omega(n^{\log_b a + \epsilon})$  for some constant  $\epsilon > 0$   
 $f(n)$  grows polynomially faster than  $n^{\log_b a}$ , by a factor  $n^\epsilon$  factor)

and  $f(n)$  satisfies the 'regularity condition' that of  $a f(n/b) \leq c f(n)$  for some constant  $c < 1$

**Solution:**  $T(n) = \Theta(f(n))$

## Examples

1.  $T(n) = 4T(n/2) + n$

here  $a=4$ ,  $b=2 \Rightarrow n^{\log_b a} = n^2$ ;  $f(n) = n$

**Case 1:**  $f(n) = O(n^{2-\epsilon})$ , for  $\epsilon=1$

$\therefore T(n) = \Theta(n^2)$

2.  $T(n) = 4T(n/2) + n^2$

$a=4$ ,  $b=2 \Rightarrow n^{\log_b a} = n^2$ ;  $f(n) = n^2$

**Case 2:**  $f(n) = \Theta(n^2 \log^0 n)$ , i.e.  $k=0$

$\therefore T(n) = \Theta(n^2 \log n)$

3.  $T(n) = 4T(n/2) + n^3$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^3$

Case 3:  $f(n) = \Omega(n^{2+\epsilon})$  for  $\epsilon = 1$

and  $4(cn/2)^2 \leq cn^3$  (reg cond) for  $c = 1/2$

$\therefore T(n) = \Theta(n^3)$

4.  $T(n) = 4T(n/2) + \frac{n^2}{\lg n}$

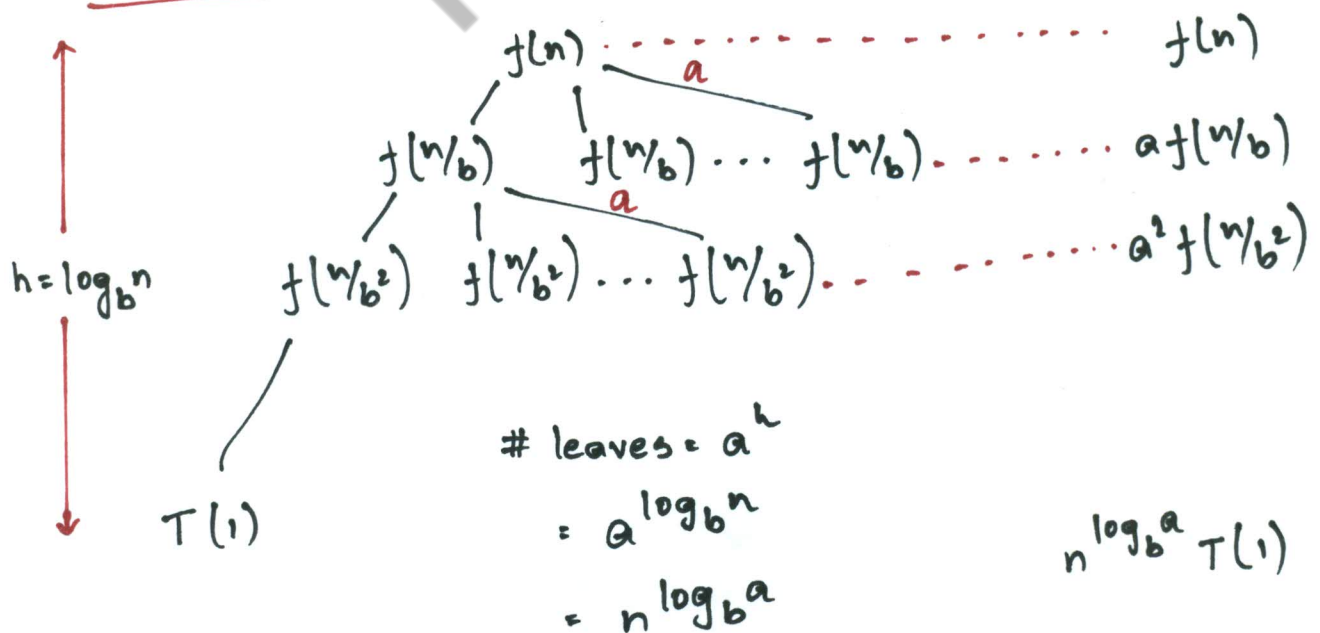
$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = \frac{n^2}{\lg n}$

Master method **does not** apply.

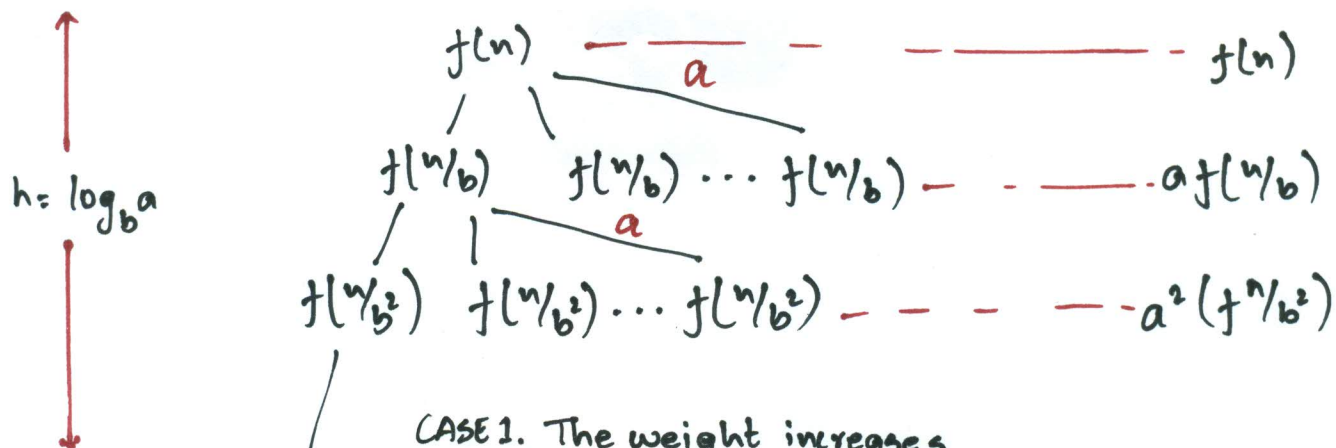
In particular for every constant  $\epsilon > 0$ , we have  $n^\epsilon = \omega(\lg n)$

## The Idea of Master Theorem

### Recursion Tree:



## Recursion Tree

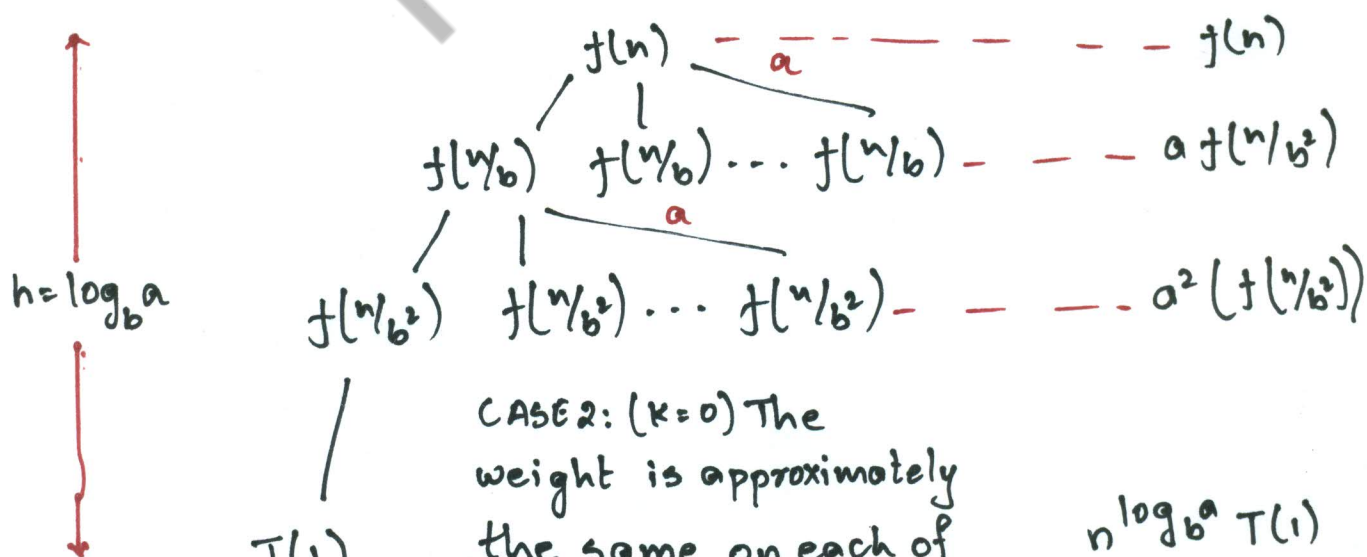


$T(1)$

CASE 1. The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight

$$\frac{n^{\log_b a} T(1)}{\Theta(n^{\log_b a})}$$

## Recursion Tree

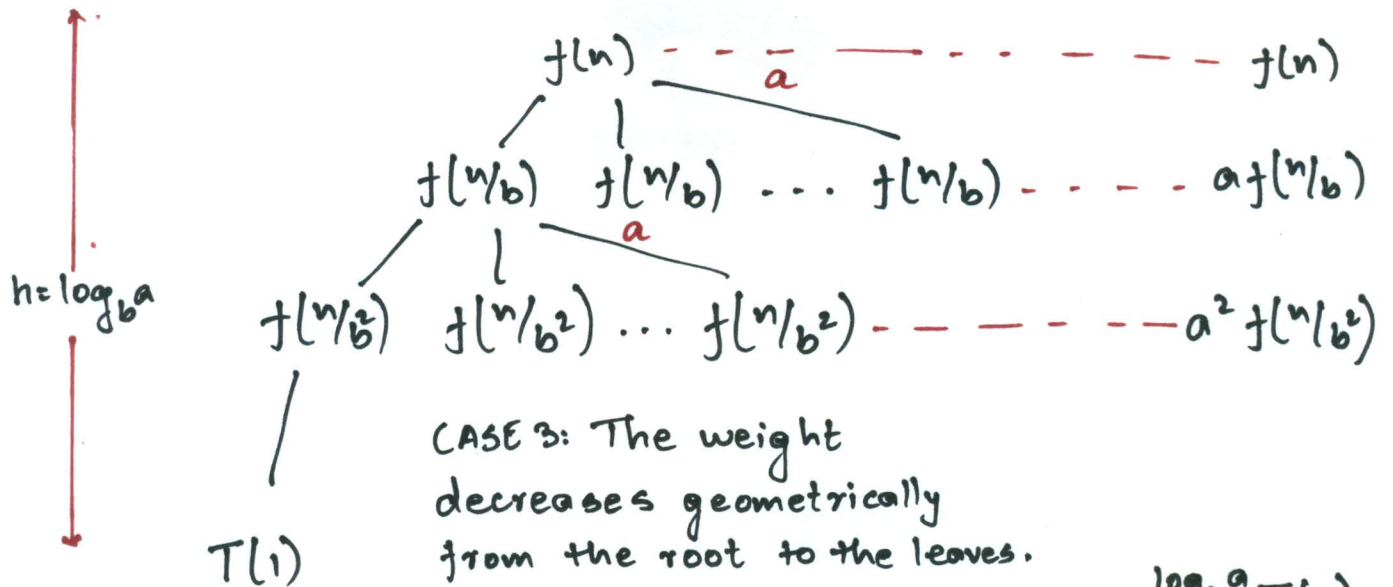


$T(1)$

CASE 2: ( $k=0$ ) The weight is approximately the same on each of the  $\log_b n$  levels

$$\frac{n^{\log_b a} T(1)}{\Theta(n^{\log_b a} \lg n)}$$

## Recursion Tree



CASE 3: The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

$$\frac{n^{\log_b a} T(1)}{\Theta(f(n))}$$



## The divide-and-conquer design paradigm

1. Divide the problem (instance) into subproblems
2. Conquer the subproblems by solving them recursively
3. Combine subproblem solutions

### Example: Merge Sort

1. Divide: Trivial
2. Conquer: Recursively sort 2 subarrays
3. Combine: Linear-time merge

$$T(n) = 2 T(n/2) + O(n)$$

# subproblems

subproblem size

work dividing and combining

## Binary search

Find an element in a sorted array

1. Divide - Check middle element
2. Conquer - Recursively search 1 subarray
3. Combine - Trivial

Example:

Find 9 in

3 5 7 8 9 12 15

3 5 7 8 9 12 15

3 5 7 8 9 12 15

3 5 7 8 9 12 15

3 5 7 8 9 12 15

3 5 7 8 9 12 15

## Recurrence for binary search

$$T(n) = 1 T(n/2) + \Theta(1)$$

$$n^{\log_b a} = n^{\log_2 1} = n^{\log_2 2} = n^0 = 1 \Rightarrow \text{CASE 2 (k=0)}$$

$$\Rightarrow T(n) = \Theta(\log n)$$

## Powering a Number

Problem: Compute  $a^n$ ,  $n \in \mathbb{N}$

Naive algorithm:  $\Theta(n)$

Divide and conquer algorithm:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd} \end{cases}$$

$$T(n) = T(n/2) + \Theta(1)$$

$$\Rightarrow T(n) = \Theta(\lg n)$$

## Fibonacci Numbers

Recursive definition:

$$F_n = \begin{cases} 0 & \text{if } n=0 \\ 1 & \text{if } n=1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2 \end{cases}$$

0 1 1 2 3 5 8 13 21 ...

Naive recursive algorithm:  $\Omega(\phi^n)$   
(exponential time), where  $\phi = (1+\sqrt{5})/2$  is  
the 'golden ratio'.



# Computing Fibonacci Numbers

Naive recursive squaring:

$$F_n = \frac{\phi^n}{\sqrt{5}}, \text{ rounded to nearest integer}$$

- Recursive squaring:  $\Theta(\lg n)$  time
- This method is unreliable, since floating-point arithmetic is prone to round-off errors.

Bottom-up:

- Compute  $F_0, F_1, F_2, \dots, F_n$  in order, forming each number by summing the two previous.
- Running time:  $\Theta(n)$

## Recursive Squaring

Theorem: 
$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$$

Algorithm: Recursive squaring

$$\text{Time} = \Theta(\lg n)$$

Proof of theorem: (Induction on  $n$ )

For  $n=1$  :  $\begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}'$ , which is true

Inductive step ( $n \geq 2$ )

$$\begin{aligned} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} &= \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \end{aligned}$$

## Matrix Multiplication

Input:  $A = [a_{ij}]$ ,  $B = [b_{ij}]$

Output:  $C = [c_{ij}] = A \cdot B$ ,  $i, j = 1, 2, \dots, n$

$$c_{i,j} = \sum_{k=1}^n a_{ik} \cdot b_{kj}, \quad i, j = 1, 2, \dots, n$$

## Standard Algorithm

1. for  $i \leftarrow 1$  to  $n$
2.     do for  $j \leftarrow 1$  to  $n$
3.         do  $c_{ij} \leftarrow 0$
4.         for  $k \leftarrow 1$  to  $n$
5.             do  $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$

Running time =  $\Theta(n^3)$

# Divide-and-Conquer Algorithm

IDEA:

$n \times n$  matrix = 2x2 matrices of  $(n/2) \times (n/2)$   
Submatrices

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C = A \cdot B$$

$$\left. \begin{array}{l} r = ae + bg \\ s = af + bh \\ t = ce + dh \\ u = cf + dh \end{array} \right\} \begin{array}{l} 8 \text{ mults of } (n/2) \times (n/2) \\ \text{submatrices} \\ 4 \text{ adds of } (n/2) \times (n/2) \\ \text{submatrices} \end{array}$$

## Analysis of Divide and Conquer algorithm

$$T(n) = 8T(n/2) + \Theta(n^2)$$

# submatrices      Submatrix size      work adding submatrices

$$n^{\log_b a} = n^{\log_2 8} = n^3 \Rightarrow \text{CASE 1: } T(n) = \Theta(n^3)$$

No better than the ordinary algorithm

## Strassen's Idea

Multiply  $2 \times 2$  matrices with only 7 recursive multiplications

$$P_1 = a \cdot (j-h)$$

$$P_2 = (a+b) \cdot h$$

$$P_3 = (c+d) \cdot e$$

$$P_4 = d \cdot (g-e)$$

$$P_5 = (a+d) \cdot (e+h)$$

$$P_6 = (b-d) \cdot (g+h)$$

$$P_7 = (a-c) \cdot (e+f)$$

$$r = P_5 + P_4 - P_2 + P_1$$

$$s = P_1 + P_2$$

$$t = P_3 + P_4$$

$$u = P_5 + P_1 - P_3 - P_2$$

7 mults, 18 adds/subs

Note: No reliance on commutativity of mults.

Here,

$$r = P_5 + P_4 - P_2 + P_1$$

$$= (a+d)(e+h) + d(g-e) - (a+b)h + (b-d)(g+h)$$

$$= ae + ah + de + dh + dg - de - ah - bh + bg + bh - dg - dh$$

$$= ae + bg$$

Similarly, others can be proved



## Strassen's Algorithm

1. Divide: Partition A and B into  $(n/2) \times (n/2)$  submatrices. Form terms to be multiplied using + and -
2. Conquer: Perform 7 multiplications of  $(n/2) \times (n/2)$  submatrices recursively
3. Combine: Form C using + and - on  $(n/2) \times (n/2)$  submatrices.

$$T(n) = 7T(n/2) + \Theta(n^2)$$

## Analysis of Strassen's Algorithm

$$T(n) = 7T(n/2) + \Theta(n^2)$$

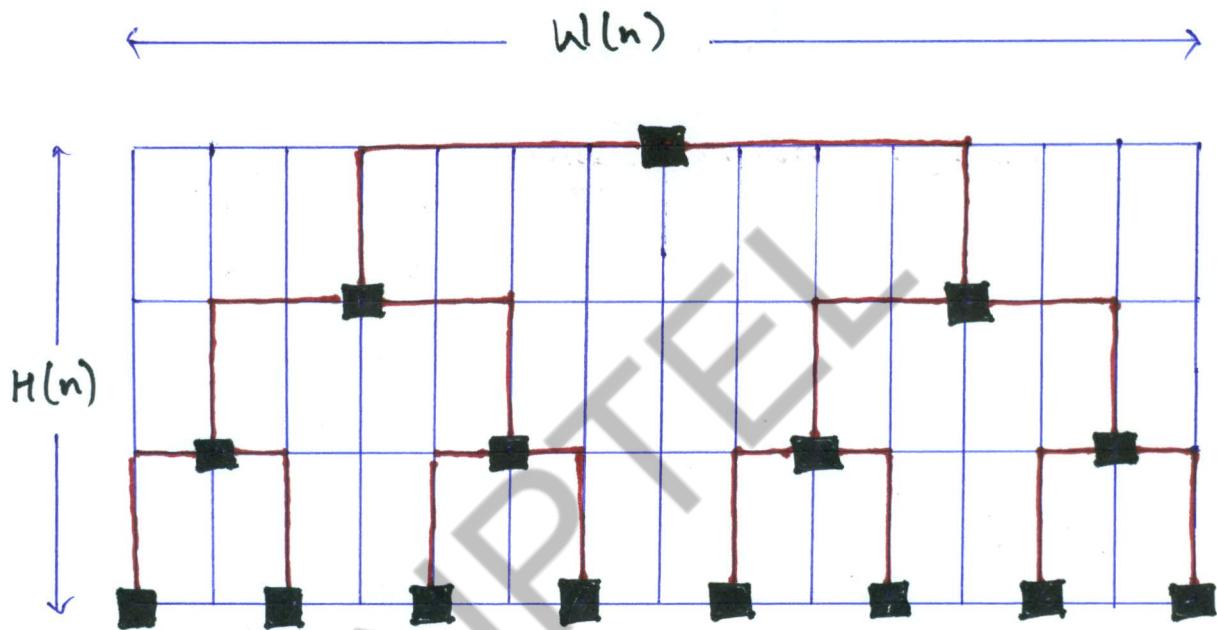
$$n^{\log_2 7} = n^{\log_2 7} \approx n^{2.81} \Rightarrow \text{CASE 1: } T(n) = \Theta(n^{\log_2 7})$$

The number 2.81 may not seem much smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for  $n \geq 30$  or so.

Best upto date:  $\Theta(n^{2.376...})$

# VLSI Layout

**Problem:** Embed a complete binary tree with 'n' leaves in a grid using minimal area.

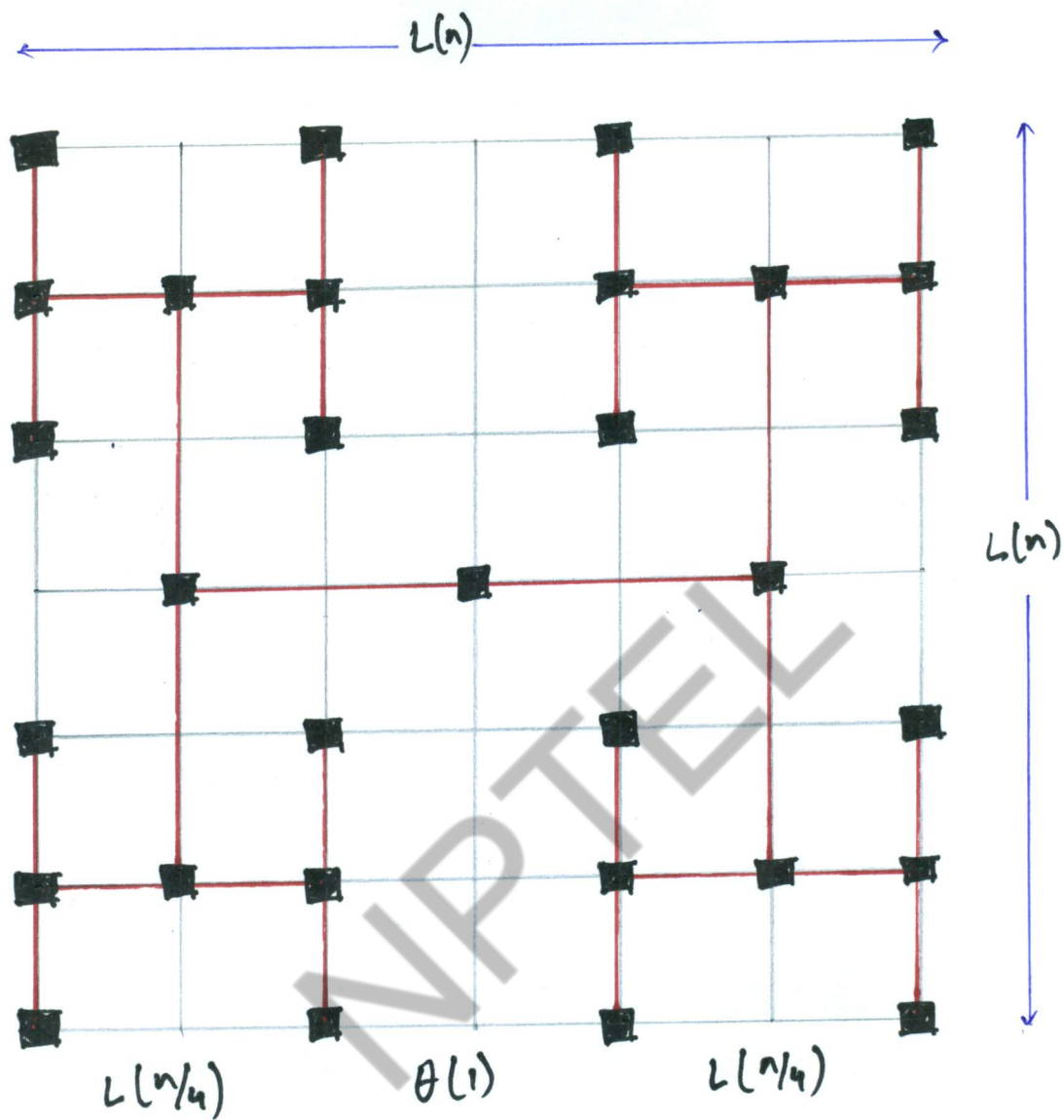


$$\begin{aligned} H(n) &= H(n/2) + \Theta(1) \\ &= \Theta(\lg n) \end{aligned}$$

$$\begin{aligned} W(n) &= 2W(n/2) + \Theta(1) \\ &= \Theta(n) \end{aligned}$$

$$\underline{\text{Area} = \Theta(n \lg n)}$$

# H-tree embedding



$$L(n) = 2L(n/4) + \theta(1)$$

$$= \theta(\sqrt{n})$$

$$\text{Area} = \theta(n)$$

H-tree embedding

## Quick Sort

- Proposed by C.A.R. Hoare in 1962
- Divide and Conquer Algorithm
- Sorts "in place" (like insertion sort)
- Very practical

## Divide and Conquer

Quicksort an  $n$ -element array :

1. Divide: Partition the array into two subarrays around a pivot  $x$  such that elements in lower subarray  $\leq x \leq$  elements in upper subarray



2. Conquer: Recursively sort the two subarrays
3. Combine: Trivial

**Key:** Linear-time partitioning subroutine

## Partitioning Subroutine

PARTITION ( $A, p, q$ )  $\triangleright A[p \dots q]$

1.  $x \leftarrow A[p] \triangleright \text{pivot} = A[p]$
2.  $i \leftarrow p$
3. for  $j \leftarrow p+1$  to  $q$
4.     do if  $A[j] \leq x$
5.         then  $i \leftarrow i+1$
6.             exchange  $A[i] \leftrightarrow A[j]$
7.     exchange  $A[p] \leftrightarrow A[i]$
8.     return  $i$

## Example of partitioning

6    10   13   5   8   3   2   11

6    5    13   10   8   3   2   11

6    5    3    10   8   13   2   11

6    5    3    2    8   13   10   11

2    5    3    6    8   13   10   11  
          ↑  
          *i*