Week 8: Lecture Notes

Topics: Dynamic programming
Longest Common Subsequence
Graphs
Prim's Algorithms
Graph Search.

Dynamic Programming

Design technique:

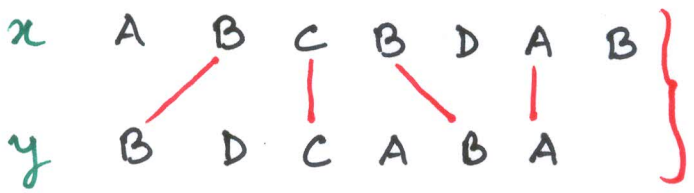
Like divide-and-conquer

Example:

Longest Common Subsequence (LCS)

Given two sequences $x[1 \ldots m]$ and $y[1 \ldots n]$, find a longest subsequence common to them both.

'a' not 'the'



BCBA = LCS(x,y)

functional notation but not a function

Brute-Force LCS Algorithm

Check every subsequence of $x[1 \ldots m]$ to see if it is also a subsequence of $y[1 \ldots n]$.

Analysis:

- Checking = $O(n)$ time per subsequence
- 2^m subsequences of x (each bit-vector of length m determines a distinct subsequence of x).

Worst case running time = $O(n 2^m)$

= exponential time

Towards a better algorithm

Simplification:

1. Look at the length of the longest-common subsequence.
2. Extend the algorithm to find the LCS of sequence s.

Notation:

- Denote the length of a sequence s by $|s|$

Strategy:

- Consider prefixes of x and y.
- Define $c[i,j] = |LCS(x[1 \ldots i], y[1 \ldots j])|$
- Then $c[m,n] = |LCS(x,y)|$.

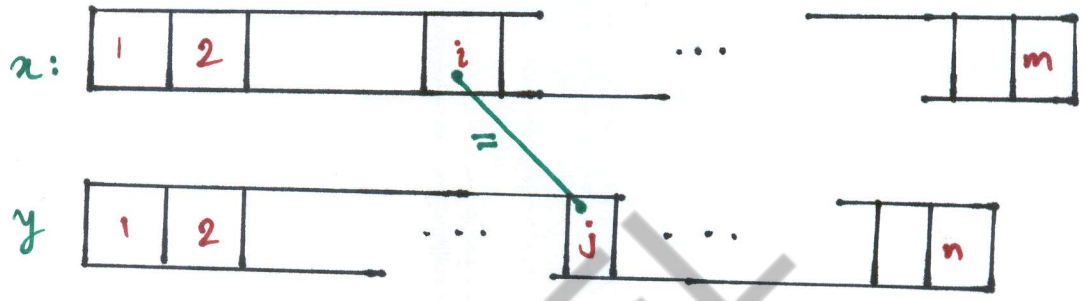
Recursive Formulation

Theorem:

$$c[i,j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j] \\ \max\{c[i-1,j], c[i,j-1]\} & \text{otherwise} \end{cases}$$

Proof:

Case $x[i] = y[j]$:



Let $z[1 \ldots k] = LCS(x[1 \ldots i], y[1 \ldots j])$, where $c[i,j] = k$.

Then, $z[k] = x[i]$, or else z could be extended.

Thus, $z[i \ldots k-1]$ is CS of $x[1 \ldots i-1]$ and $y[1 \ldots j-1]$

Claim: $z[1 \ldots k-1] = LCS(x[1 \ldots i-1], y[1 \ldots j-1])$.

Suppose w is longer CS of $x[1 \ldots i-1]$ and $y[1 \ldots j-1]$, that is $|w| > k-1$.

Then cut and paste: $w \| z[k]$ (w concatenated with $z[k]$) is a common subsequence of x and y with

$$|w \| z[k]| > k.$$

Contradiction, proving claim.

Thus $c[i-1, j-1] = k-1$, which implies that $c[i,j] = c[i-1, j-1] + 1$.

Other cases are similar.

Dynamic Programming hallmark #1

Optimal Substructure:

An optimal solution to a problem (instance) contains optimal solutions to subproblems.

- If $z = LCS(x,y)$, then any prefix of z is an LCS of a prefix of x and a prefix of y.

Recursive Algorithm for LCS

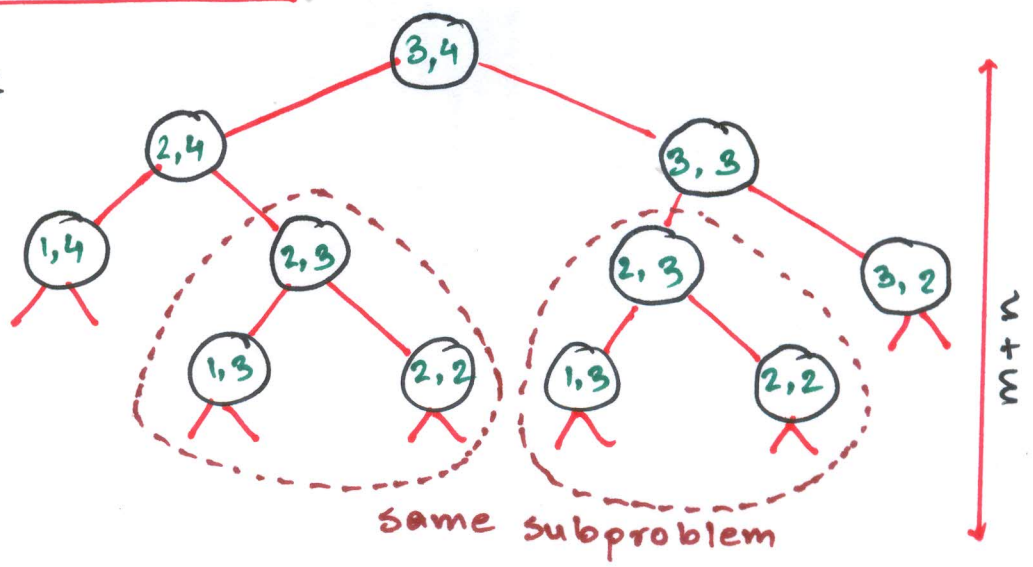
LCS (x, y, i, j)

1. if $x[i] = y[j]$
2. then $c[i,j] \leftarrow LCS(x,y,i-1,j-1) + 1$
3. else $c[i,j] \leftarrow \max\{LCS(x,y,i-1,j), LCS(x,y,i,j-1)\}$

Worst case: $x[i] \neq y[j]$, in which case the algorithm evaluates two subproblems, each with only one parameter decremented.

Recursion Tree

$m = 3$
$n = 4$:



Height = $m+n$ $\Rightarrow$ work potentially exponential, but we are solving subproblems already solved.

Dynamic Programing hallmark #2

Overlapping subproblems:

A recursive solution contains a "small" number of distinct subproblems repeated many times.

- The number of distinct LCS subproblems for two strings of lengths m and n is only mn.

Memoization Algorithm

Memoization:

After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

```
LCS(x,y,i,j)
    if c[i,j] = NIL
       then if x[i] = y[i]
                then c[i,j] ← LCS(x,y,i-1,j-1) + 1
                else c[i,j] ← {LCS(x,y,i-1,j),
                               LCS(x,y,i,j-1)}
```

} same as before

Time: $\Theta(mn)$ = constant work per table entry

Space: $\Theta(mn)$

Graphs (review)

Definition

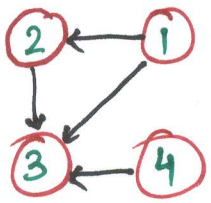
- A directed graph (digraph), $G = (V, E)$ is an ordered pair consisting of:
  - a set V of vertices (singular vertex)
  - a set $E \subseteq V \times V$ of edges.
- In an undirected graph, $G = (V, E)$, the edge set E consists of unordered pair of vertices.
- In either case, we have $|E| = O(V^2)$
- Moreover if G is connected, then $|E| \geq |V| - 1$, which implies that $\log |E| = \Theta(\log V)$

Adjacency matrix representation

The adjacency matrix of a graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, is the matrix $A[1 \ldots n, 1 \ldots n]$ given by

$$A[i,j] = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \end{cases}$$

Example.



A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

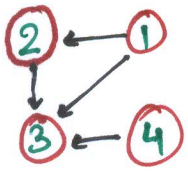
$O(V^2)$ storage

$\Rightarrow$ dense representation.

Adjacency - list representation

An adjacency list of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to v.

Example:



$Adj[1] = \{2, 3\}$

$Adj[2] = \{3\}$

$Adj[3] = \{\}$

$Adj[4] = \{3\}$

For undirected graphs:

$|Adj[v]| = degree(v)$

For digraphs, $|Adj[v]| = out\text{-}degree(v)$

Handshaking Lemma:

$\sum_{v \in V} = 2|E|$ for undirected graphs

$\Rightarrow$ adjacency list • use $\Theta(V+E)$ storage

(a sparse representation for either type of graph)

Minimum Spanning Tree (MST)

Input: A connected, undirected graph $G = (V, E)$ with weight function $w: E \to R$.

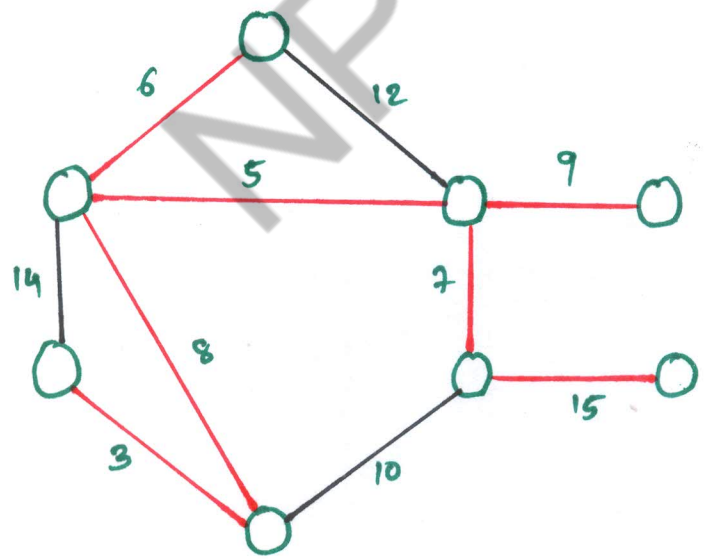
- for simplicity, assume that all edge weights are distinct.

Output:

A spanning tree T - a tree that connects all vertices - of minimum weight.

$$w(T) = \sum_{(u,v) \in T} w(u,v)$$

Example of MST

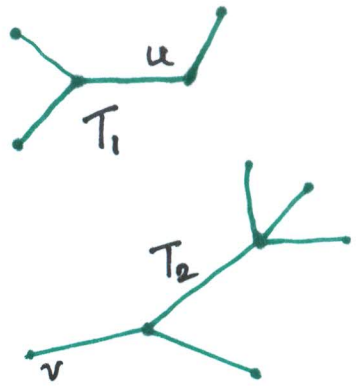


Optimal Substructure

Consider MST T of graph G.

Remove any edge $(u,v) \in T$.

Then T is partitioned into two subtrees T_1 and T_2. (other edges of G are not shown)



Theorem:

The subtree T_1 is an MST of $G_1 = (V_1, E_1)$, the graph induced by the vertices of T_1:

V_1 = vertices of T_1

$E_1 = \{(x,y) \in E : x,y \in V_1\}$

Similarly for T_2.

Proof:

Cut and paste:

$$w(T) = w(u,v) + w(T_1) + w(T_2)$$

If T_1' were a lower-weight spanning tree than T_1 for G_1, then $T' = \{(u,v)\} \cup T_1' \cup T_2$ would be a lower-weight spanning tree than T for G.

Do we also have overlapping subproblems?

- Yes
  - Great, then dynamic programming may work
- Yes, but MST exhibits another powerful property which leads to an even more efficient algorithm.

Hallmark for Greedy Algorithm

Greedy Choice Property:

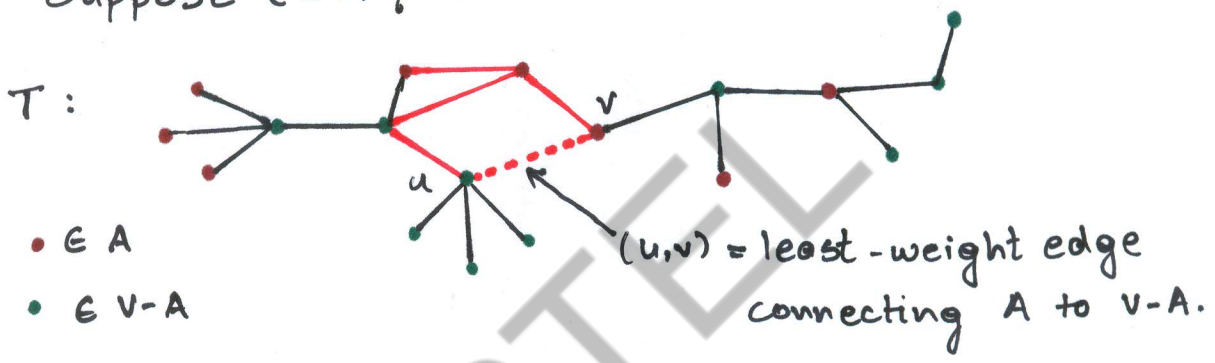
A locally optimal choice is globally optimal

Theorem:

Let T be the MST of $G=(V,E)$, and let $A \subseteq V$. Suppose that $(u,v) \in E$ is the least-weight edge connecting A to $V-A$. Then $(u,v) \in T$.

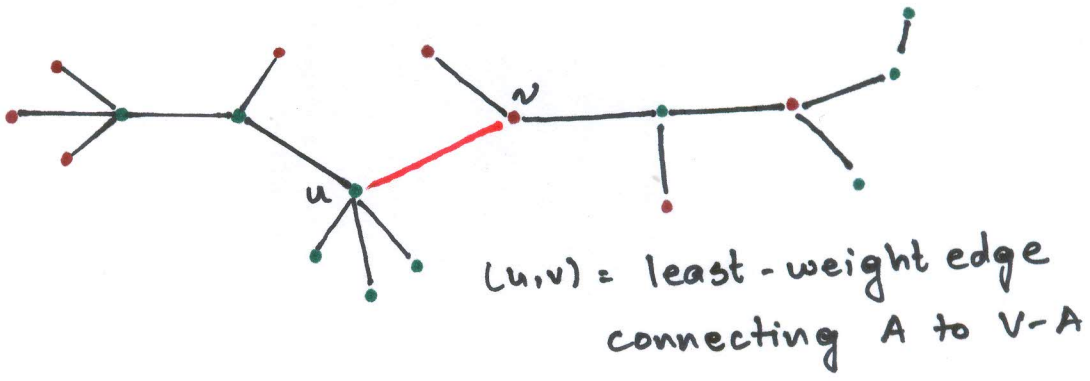
Proof:

Suppose $(u,v) \notin T$.



Consider the unique simple path from u to v in T

Swap (u,v) with the first edge on this path that connects a vertex in A to a vertex in $V-A$.



A lighter-weight spanning tree than T results.

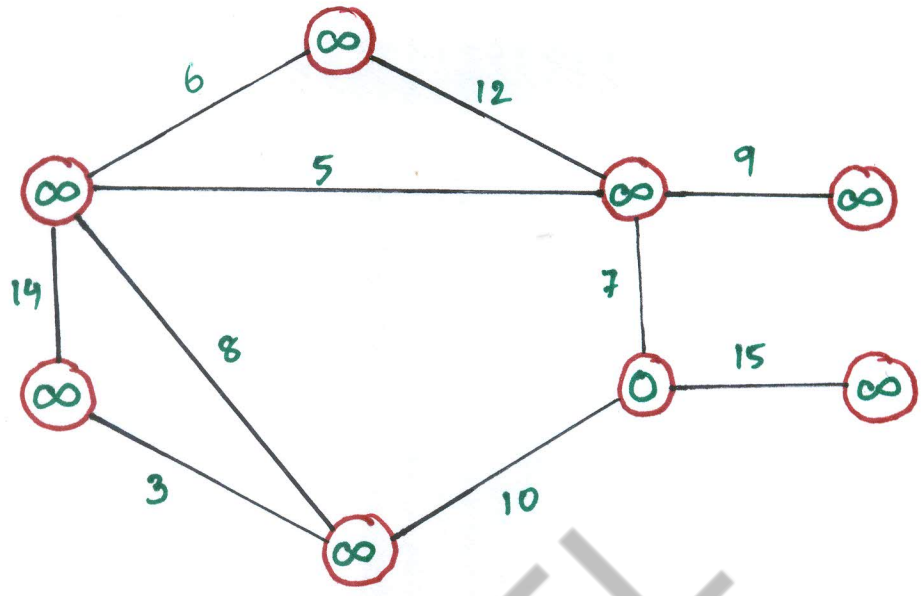
Prim's Algorithm

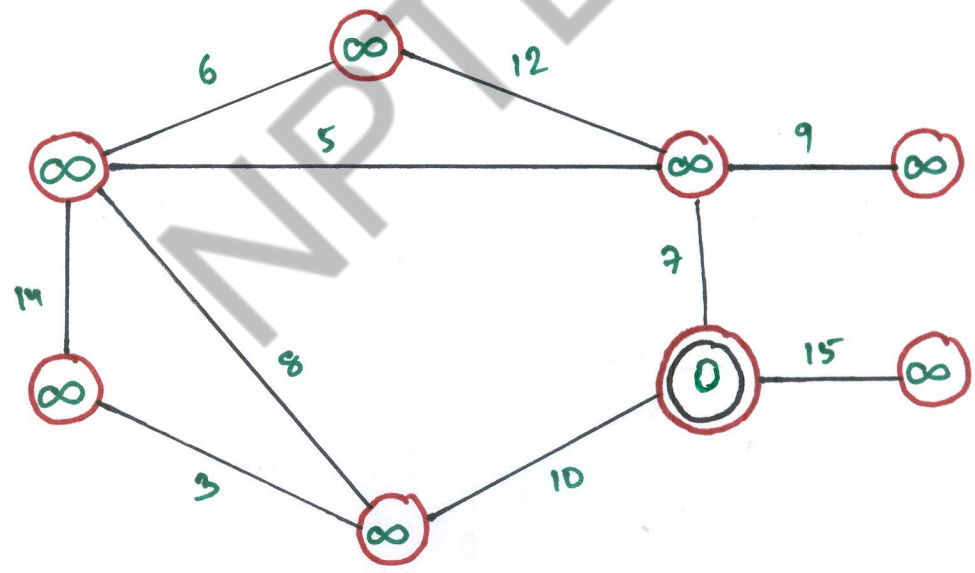
Idea: Maintain V-A as a priority queue Q. Key each vertex in Q with the weight of least-weight edge connecting it to a vector in A.

```
1.  Q ← V
2.  key[v] ← ∞ for all v ∈ V
3.  key [s] ← 0 for some arbitrary s ∈ V
4.  while Q ≠ ϕ
5.        do u ← EXTRACT-MIN (Q)
6.           for each v ∈ Adj [u]
7.                do if v ∈ Q and w(u,v) < key [u]
8.                   then key [v] ← w(u,v)
9.                        π [v] ← u
```

At the end $\{v, \pi[v]\}$ forms the MST.

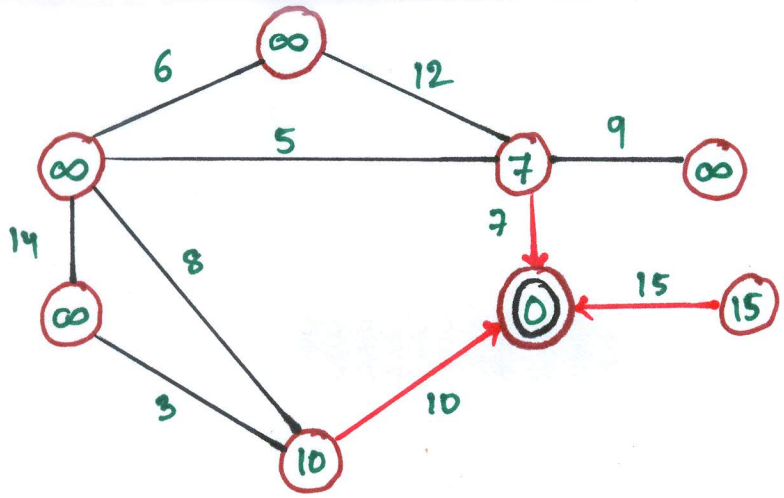
Example of Prim's Algorithm





 $\in A$

 $\in V-A$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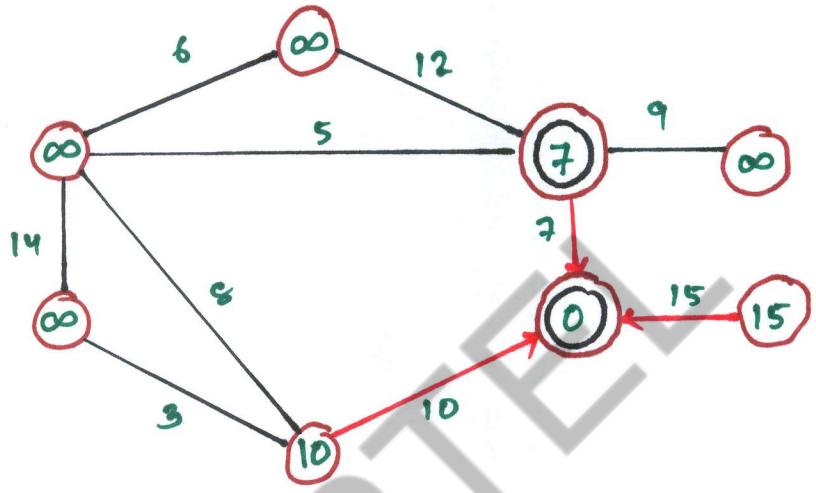
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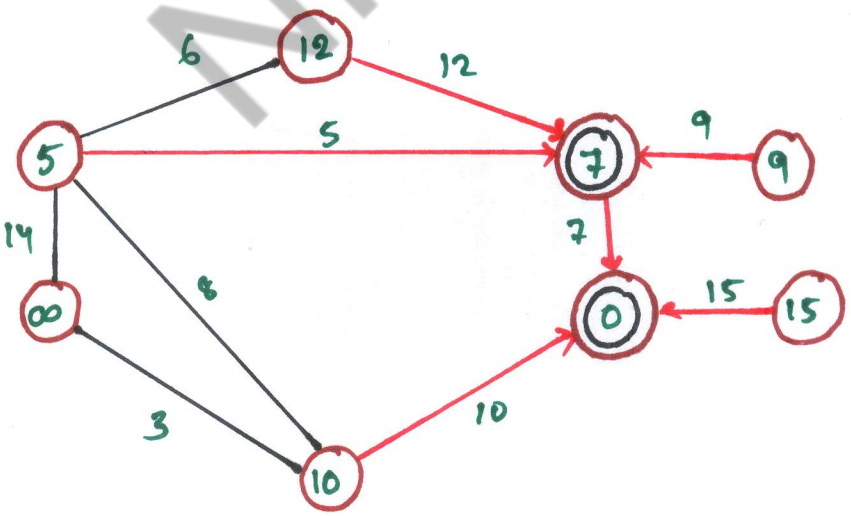
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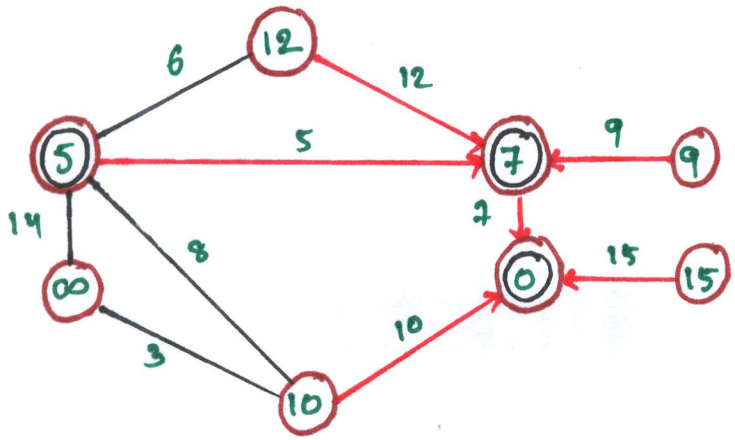
∈ A


∈ V-A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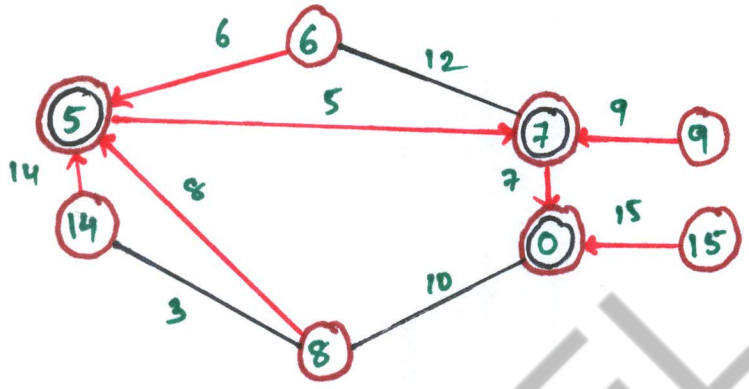
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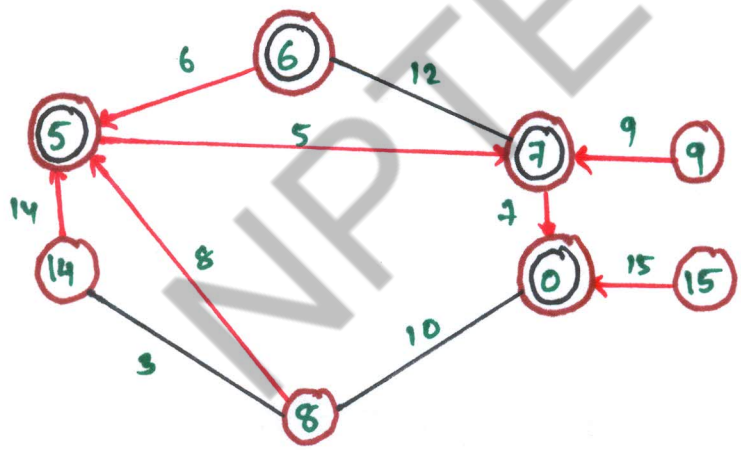
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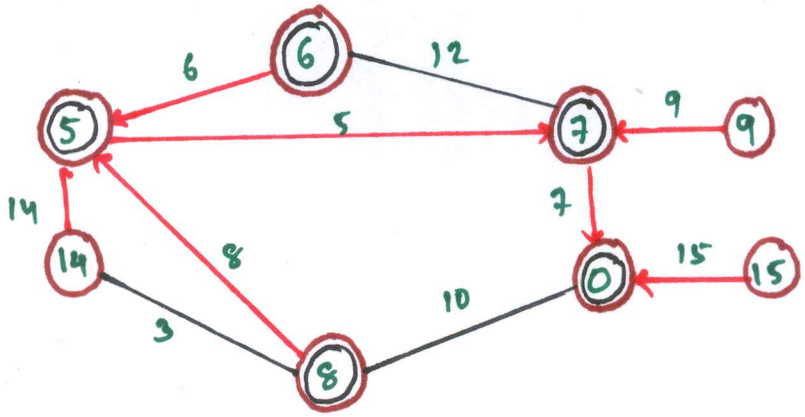
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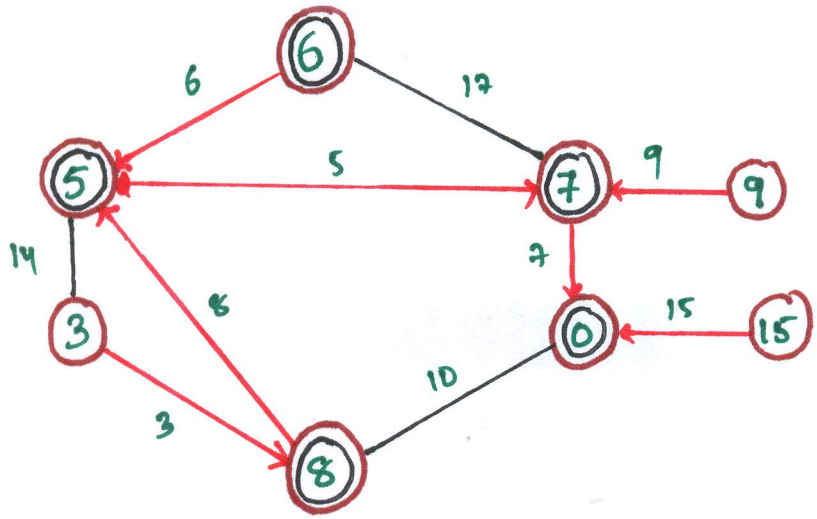
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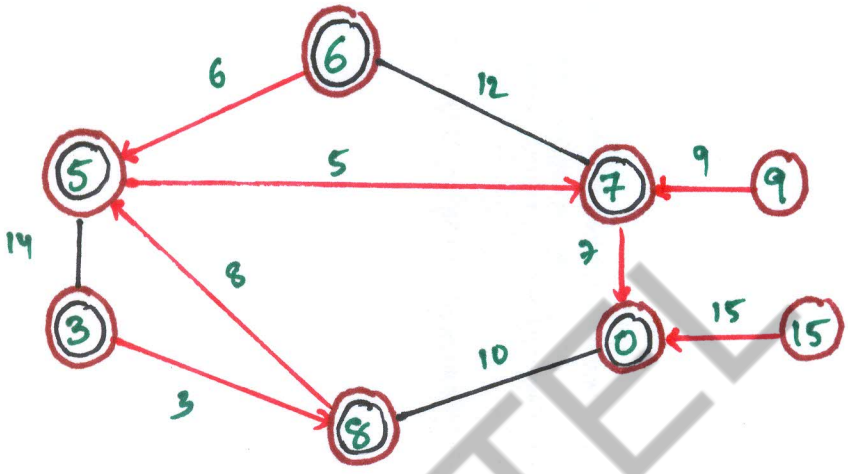
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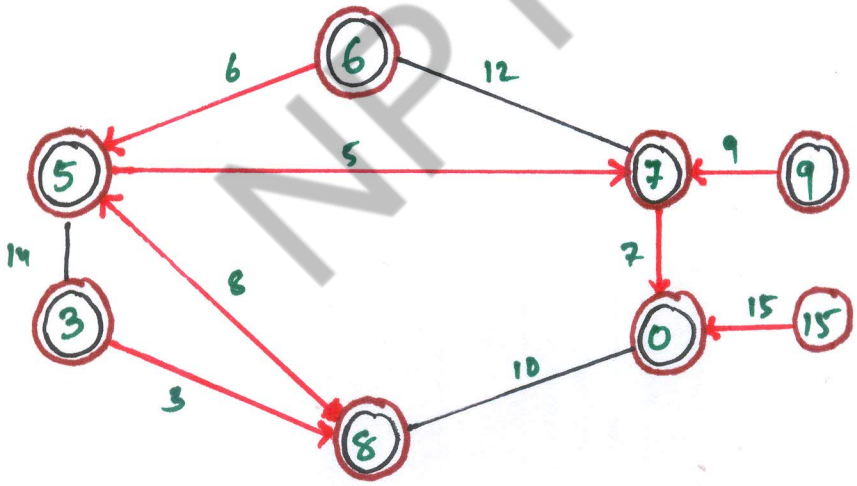
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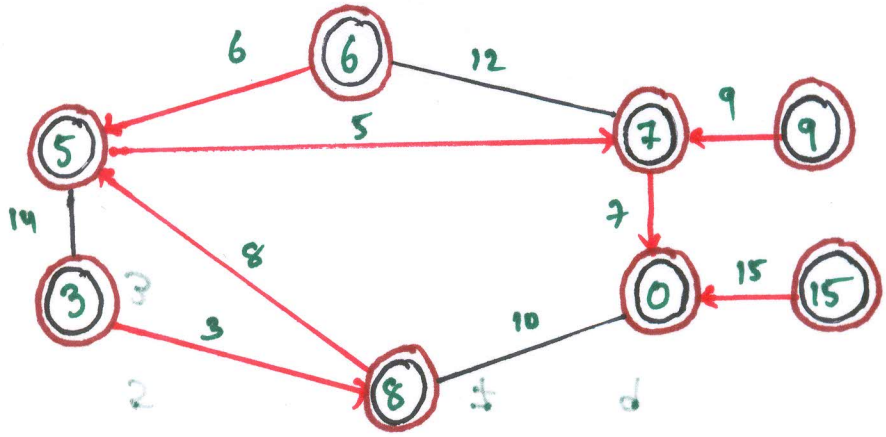
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Analysis of Prim

$O(V)$ total:
- $Q \leftarrow V$
- $key[v] \leftarrow \infty$ for all $v \in V$
- $key[s] \leftarrow 0$ for some arbitrary $s \in V$

$|V|$ times:
- while $Q \neq \phi$
- do $u \leftarrow$ EXTRACT-MIN (Q)

degree (u) times:
- for each $v \in Adj[u]$
- do if $v \in Q$ and $w(u,v) < key[v]$
- then $key[v] \leftarrow w(u,v)$
- $\pi[v] \leftarrow u$

Handshaking lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's

Time = $\Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\log V)$	$O(\log V)$	$O(E \log V)$
fibonacci heap	$O(\log V)$ amortized	$O(1)$ amortized	$O(E + V \log V)$ worst case

MST Algorithms

Kruskal's algorithm:

- Uses the disjoint-set data structure
- Running time = $O(E \log V)$

Best to date:

- Karger, Klein, and Tarjan [1993]
- Randomized algorithm
- $O(V+E)$ expected time.

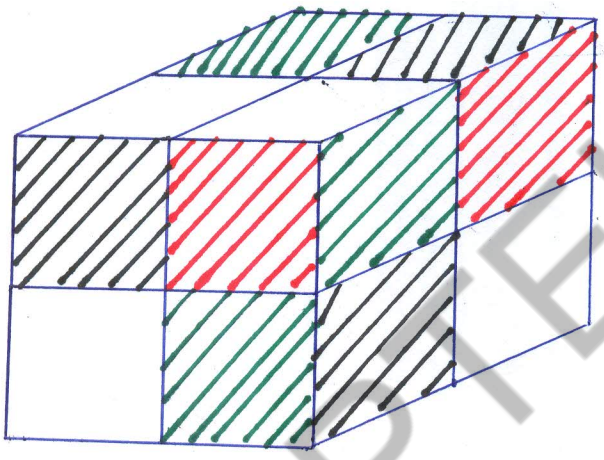
Graph Search

"Explore a graph", e.g.:

- find a path from start vertex s to a desired vertex
- visit all vertices or edges of graph, or only those reachable from s.

Pocket Cube

Consider a 2×2×2 Rubik's cube

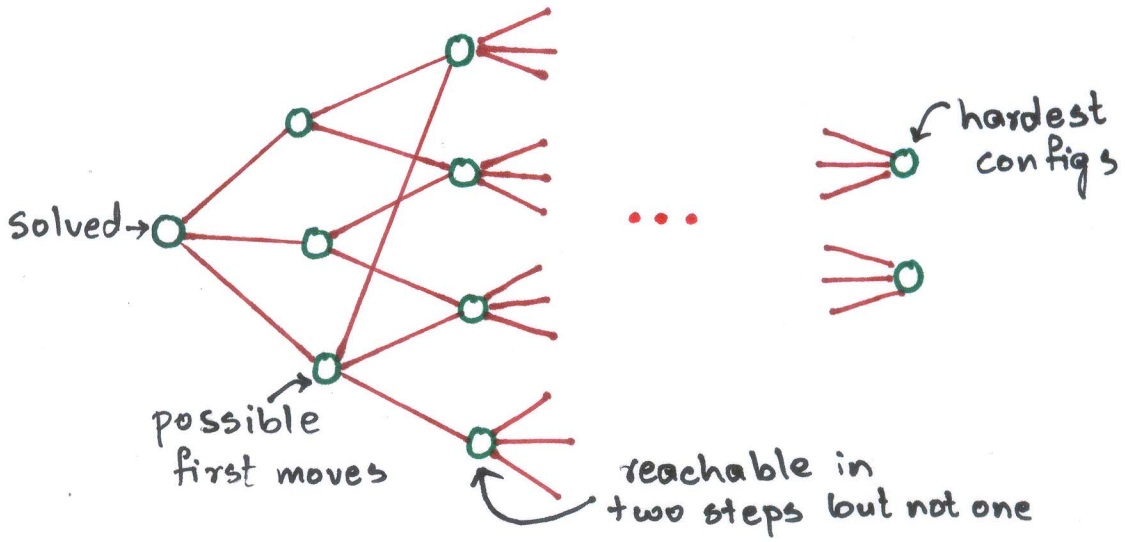


Configuration graph:

vertex for each possible state

edge for each basic move (e.g. 90° turn) from one state to another.

undirected: moves are reversible.

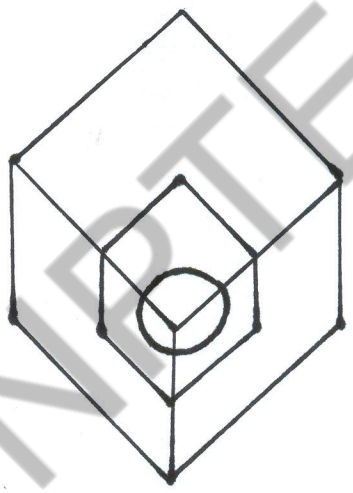


Diameter ("God's Number")

- 11 for 2×2×2
- 20 for 3×3×3
- $\Theta(n^2/\log n)$ for n×n×n [Demaine, Demaine, Eisenstat, Lubiw Winslow 2011]

- Number of vertices = $8! \cdot 3^8 = 264,539,520$

  where 8! comes from having 8 cubelets in arbitrary positions and 3^8 comes ~~from~~ as each cubelethas 3 possible twists.



This can be divided by 24 if we remove cube symmetries and further divided by 3 to account for actually reachable configurations (there are 3 connected components).