Department of Civil Engineering
IIT Madras

Photograph of the interior of the 12th-century Hoysala Temple in Halebidu, Karnataka

Failure Theories

Modern Construction Materials – Lecture 8
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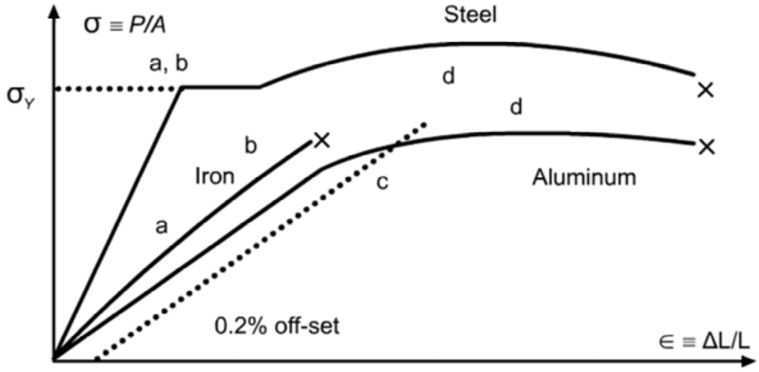
Failure of a Structural Material

- A structural material can be considered to have failed when it can no longer perform its (mechanical) design function.
- Failure generally occurs due to complete fracture (i.e., brittle failure) or excessive deformation, which may or may not result in rupture (ductile failure).
- Under uniaxial loading, the stress-strain curve can be used to represent the response until failure.
- Under multiaxial stresses, failure theories are needed for representing the material behaviour based on plasticity (or yielding) and fracture.

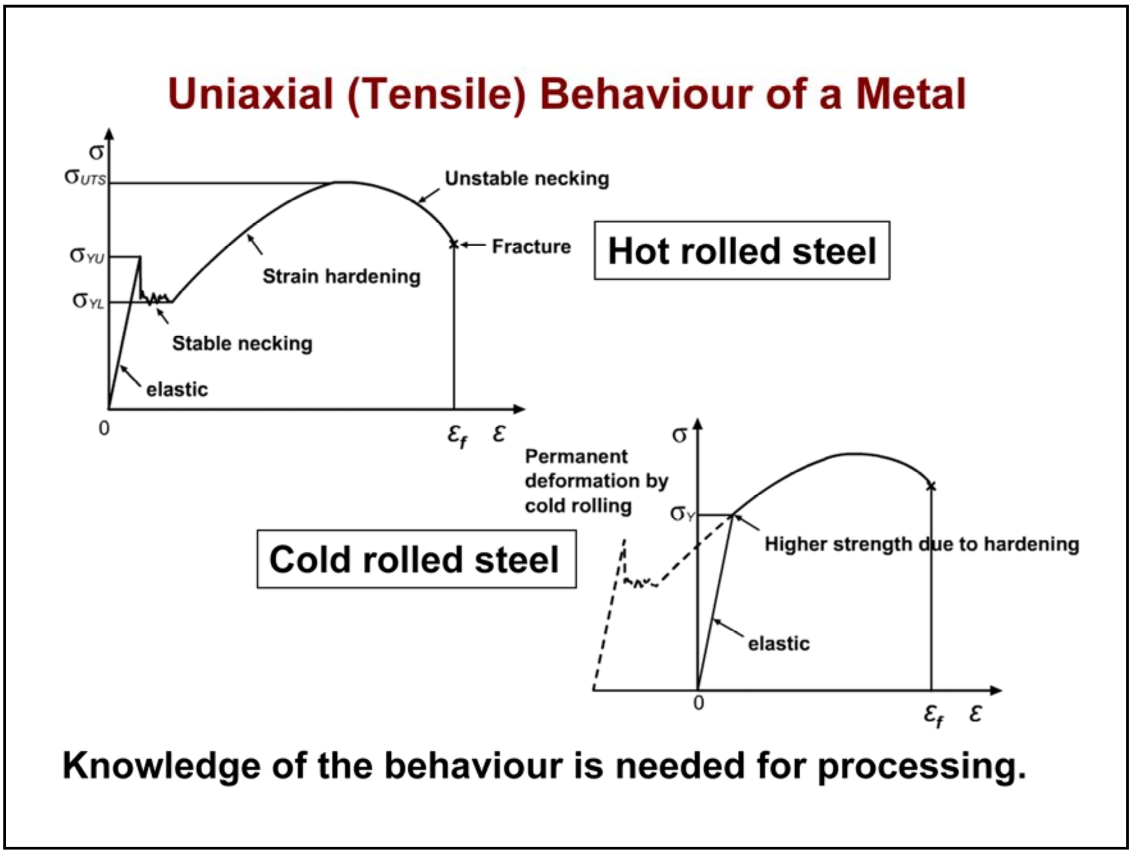
Uniaxial (Tensile) Behaviour of a Metal



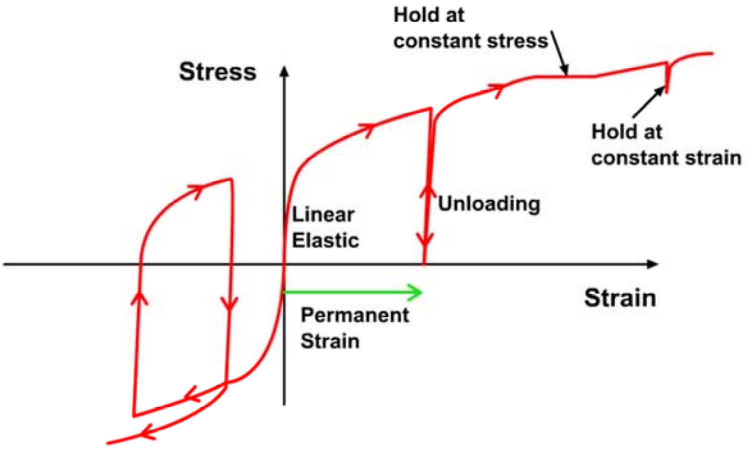
a – proportionality limit
b – elastic limit
c – yield point for 0.2% strain

d – maximum stress
× – rupture

Behaviour can vary significantly between materials.

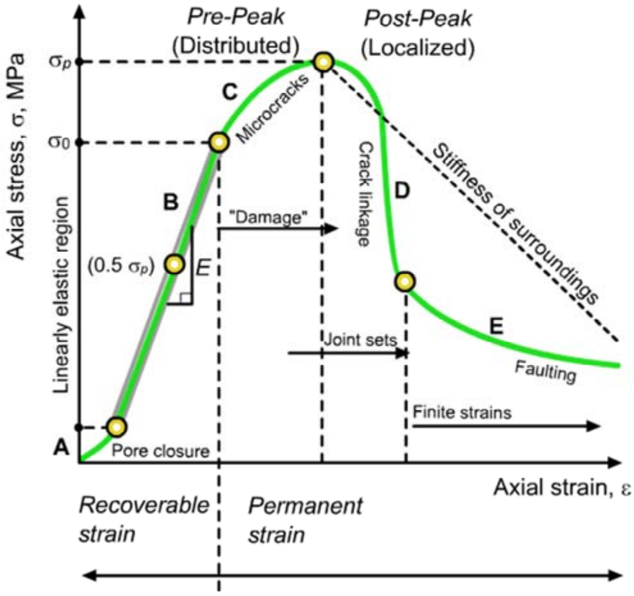

Uniaxial (Tensile) Behaviour of a Metal
σ
σUTS
σYU
σYL
Unstable necking
Fracture
Strain hardening
Stable necking
elastic
0
εf
ε
Hot rolled steel
σ
Permanent
deformation by
cold rolling
σY
Higher strength due to hardening
elastic
0
εf
ε
Cold rolled steel
Knowledge of the behaviour is needed for processing.

Complex Inelastic Response: *Metals*

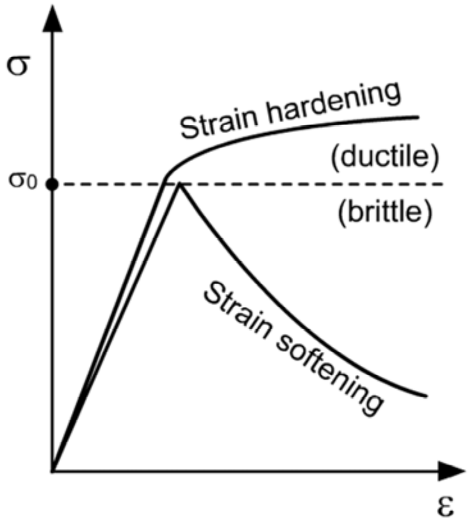


Modelling the complete behaviour is often not possible since many different mechanisms have to be considered.

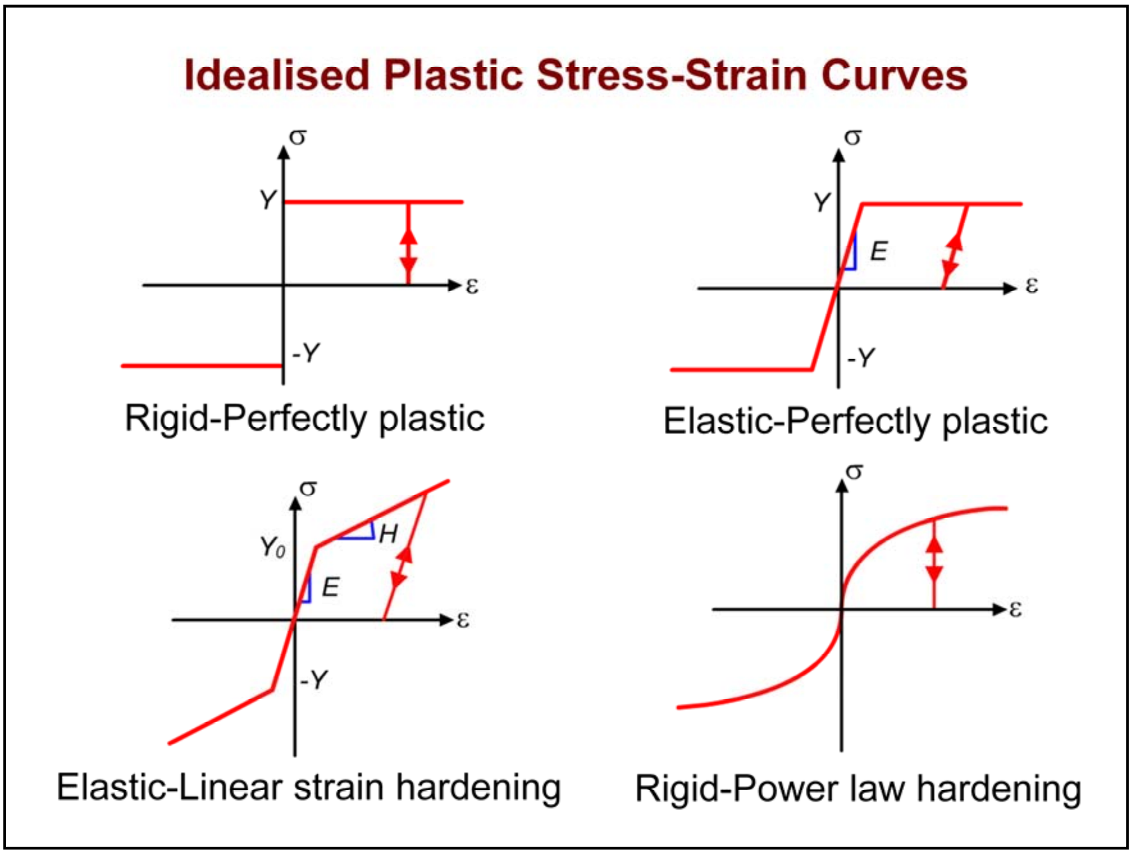
Complex Inelastic Response: *Rock, Concrete*



Complex Inelastic Response: *Rock, Concrete*



Materials such as concrete, rock and ceramics exhibit some nonlinearity followed by a post-peak drop in stress. This phenomenon is called *(strain) softening*.


Idealised Plastic Stress-Strain Curves
σ
Y
ε
-Y
Rigid-Perfectly plastic
σ
Y
E
ε
-Y
Elastic-Perfectly plastic
σ
Y0
H
E
ε
-Y
Elastic-Linear strain hardening
σ
ε
Rigid-Power law hardening

Multiaxial Loading

- In every stress state, there are three *principal directions*, at right angles to each other, along which the *principal stresses* act.
- Each principal stress represents the maximum (or minimum) normal stress for one set of plane stresses.
- No shearing stresses act on the principal planes.

Multiaxial Loading

- The three principal stresses are usually designated as $\sigma_1 \geq \sigma_2 \geq \sigma_3$ (where tension is considered positive).

- The maximum shear stress in a body is given by:

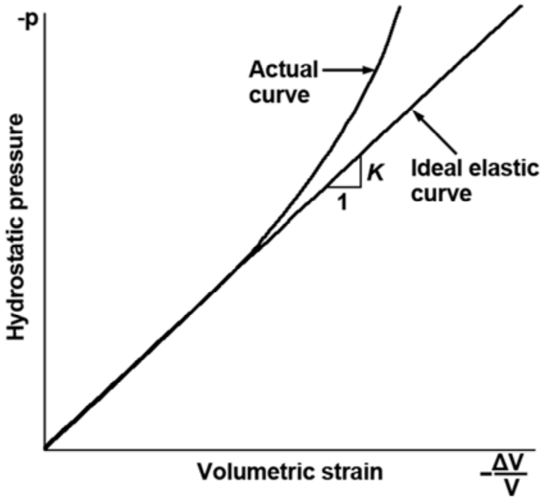
$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}$$

Multiaxial Loading: *Hydrostatic Stresses*

- Hydrostatic compression occurs when the pressure in a body is the same in all directions and is always normal to any surface on which it acts.
- No shearing stresses are possible.
- Therefore, $\sigma_1 = \sigma_2 = \sigma_3 = -p$

Multiaxial Loading: *Hydrostatic Stresses*

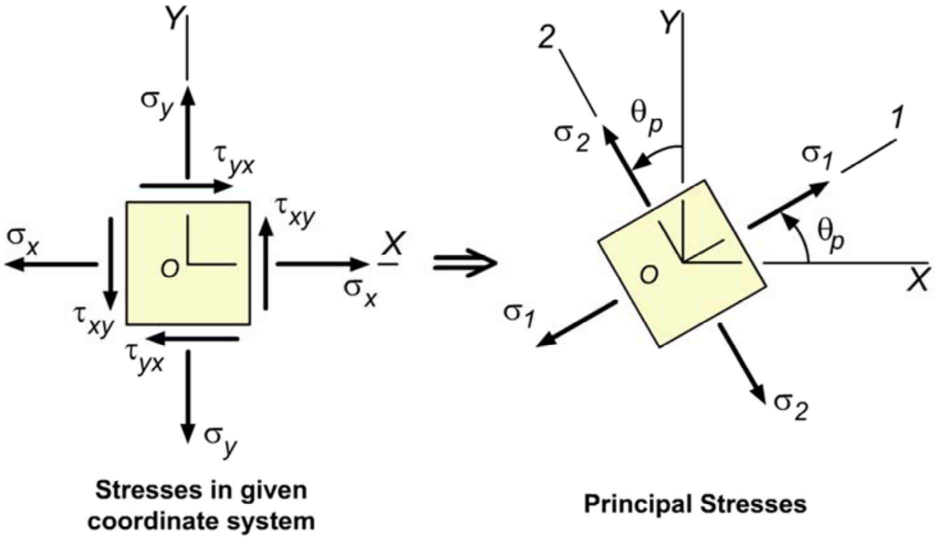
As pressure is increased, the atoms are pushed closer together and the repulsive forces increase. At higher pressure, the pressure versus volumetric strain curve exhibits an upward curvature due to the nonlinearity of the interatomic bond.



For some materials, the curve may turn downward due to the collapse of open structures (e.g., porous or network materials).

Young et al.

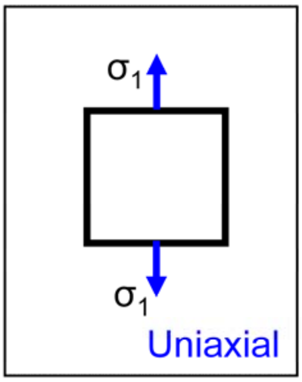
Multiaxial Loading: *Biaxial Stress State*

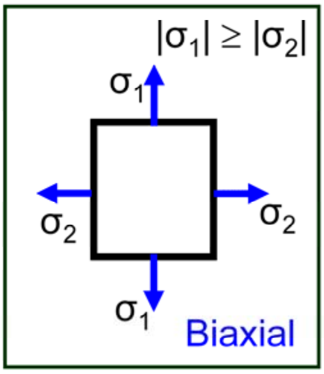


Biaxial stress states ($\sigma_3 = 0$) often occur in thin plates.

Maximum Principal Stress Criterion: *Rankine Theory*

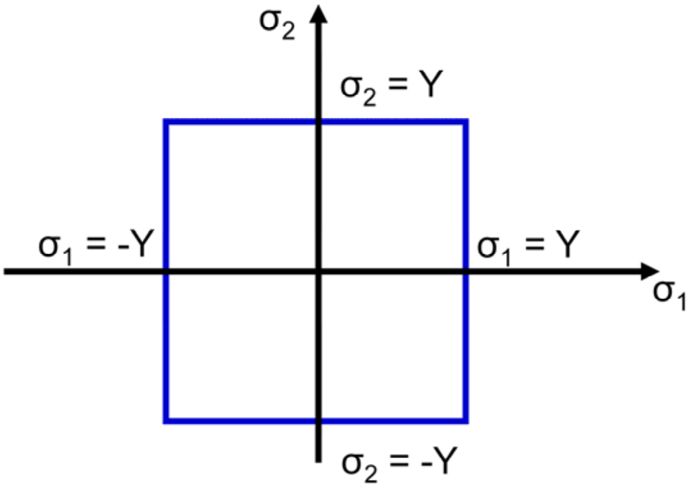
Yielding begins when the maximum principal stress reaches a value equal to the tensile (or compressive) yield stress (Y) in uniaxial tension (or compression).





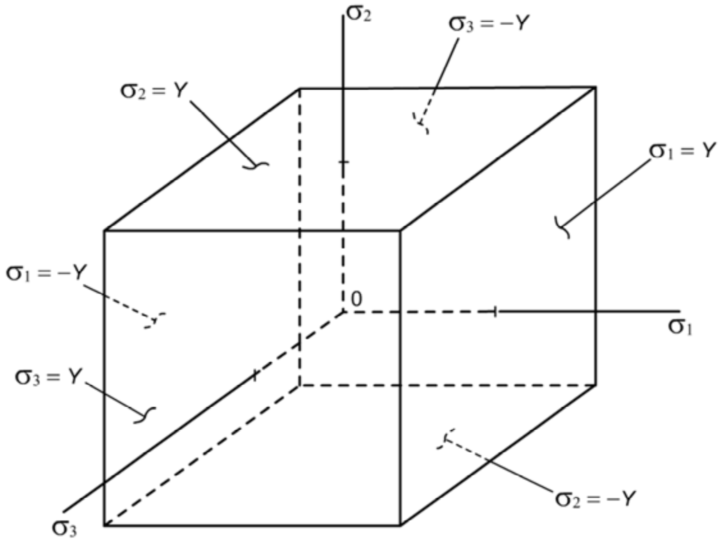
Yielding occurs when σ_1 = Y

Maximum Principal Stress Criterion: *Rankine Theory*



Maximum Principal Stress Yield Surface

Maximum Principal Stress Criterion: *Rankine Theory*



Maximum principal stress yield surface.

Maximum Shear Stress Criterion: *Tresca Criterion*

- This theory is based on the observation that in ductile materials, slip occurs during yielding, suggesting that the maximum shear stress plays the key role.

- Yielding occurs when the maximum shear stress reaches the value of the maximum shear stress at yield in uniaxial tension.

Maximum Shear Stress Criterion: *Tresca Criterion*

Uniaxial loading

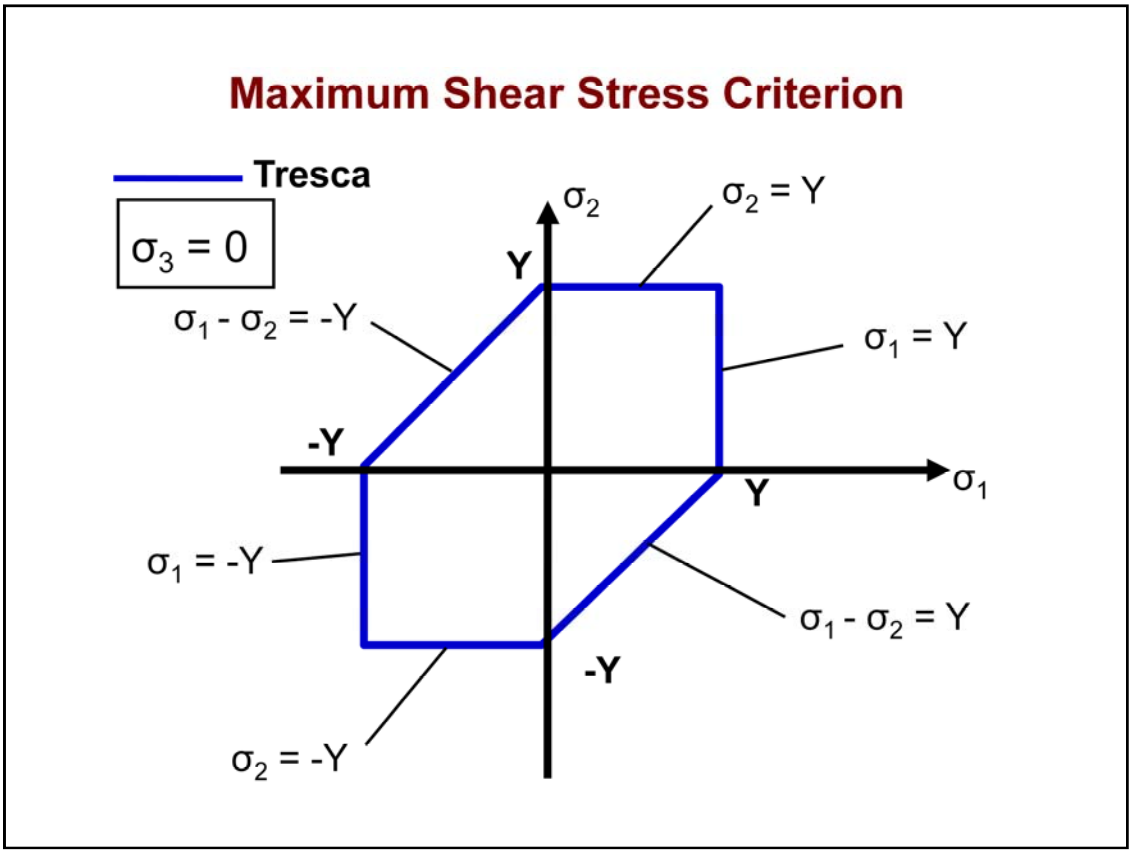
$$\sigma_1 = Y$$
$$\sigma_2 = 0$$
$$\sigma_3 = 0$$
$$\tau_{max} = \frac{Y-0}{2} = \frac{Y}{2}$$

Multiaxial loading

$$\tau_1 = \frac{|\sigma_2 - \sigma_3|}{2}$$
$$\tau_2 = \frac{|\sigma_3 - \sigma_1|}{2}$$
$$\tau_3 = \frac{|\sigma_1 - \sigma_2|}{2}$$
$$\tau_{max} = \max(\tau_1, \tau_2, \tau_3)$$

Tresca criterion

$$\sigma_2 - \sigma_3 = \pm Y$$
$$\sigma_3 - \sigma_1 = \pm Y$$
$$\sigma_1 - \sigma_2 = \pm Y$$


Maximum Shear Stress Criterion
Tresca
$\sigma_3 = 0$
σ_2
$\sigma_2 = Y$
Y
$\sigma_1 - \sigma_2 = -Y$
$\sigma_1 = Y$
-Y
σ_1
Y
$\sigma_1 = -Y$
$\sigma_1 - \sigma_2 = Y$
-Y
$\sigma_2 = -Y$

Maximum Shear Stress Criterion

- The Tresca yield criterion gives good agreement with experimental results for ductile materials. Since it is simple, it is the most often used yield theory.

- The main objection to this theory is that it ignores the effect of the intermediate principal stress. Nevertheless, only the maximum distortional strain energy theory predicts yielding better than the Tresca theory; the differences are rarely more than 15%.

Maximum Distortional Strain Energy Theory: *von Mises Theory*

- This theory is also referred to as the octahedral shear stress theory or the Huber-Hencky-von Mises theory.
- Yielding occurs when the distortional energy density reaches a value equal to the distortional energy density at yield in a uniaxial case.
- The total strain energy can be divided into two parts: the volumetric energy and the distortional energy

$$U_0 = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right)\right]$$

$$U_0 = U_V + U_D$$

Maximum Distortional Strain Energy Theory: *von Mises Theory*

- The volumetric energy (U_V) is related to the volume change under mean hydrostatic pressure.
- The distortional energy (U_D) is related to the change in shape.

$$U_V = \frac{(\sigma_1 + \sigma_2 + \sigma_3)^2}{18K}$$

$$U_D = \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{12G}$$

Bulk Modulus

$$K = \frac{E}{3(1-2\nu)}$$

Shear Modulus

$$G = \frac{E}{2(1+\nu)}$$

Maximum Distortional Strain Energy Theory

Uniaxial loading

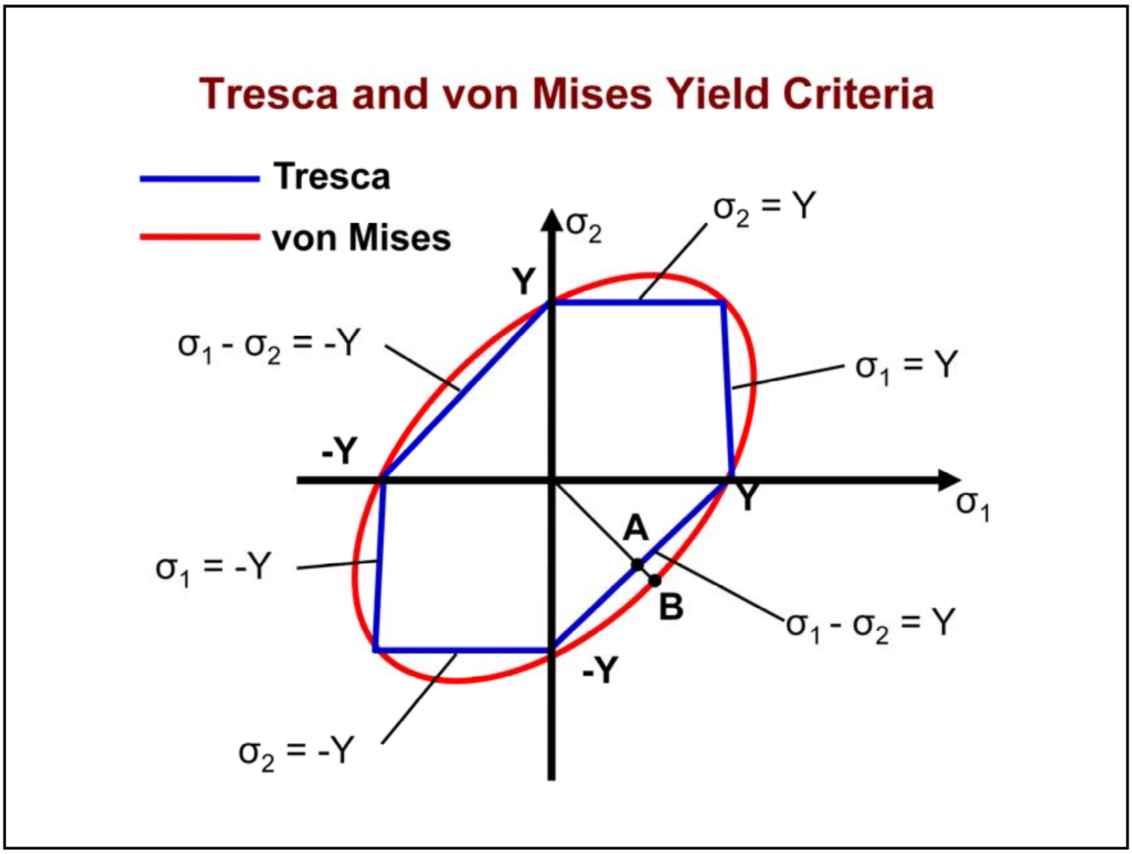
$$\sigma_1 = Y \qquad \sigma_2 = 0 \qquad \sigma_3 = 0$$

$$U_D = \frac{(Y-0)^2 + (0-0)^2 + (0-Y)^2}{12G} \qquad U_D = \frac{Y^2}{6G}$$

Multiaxial loading: **von Mises Yield Criterion**

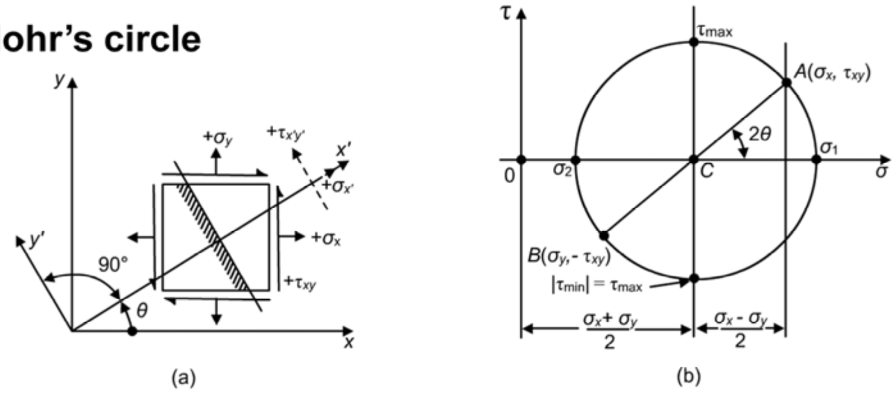
$$U_D = \frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{12G} = \frac{Y^2}{6G}$$

$$(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2$$


Tresca and von Mises Yield Criteria
Tresca
von Mises
σ_2
$\sigma_2 = Y$
Y
$\sigma_1 - \sigma_2 = -Y$
$\sigma_1 = Y$
-Y
Y
σ_1
A
$\sigma_1 = -Y$
B
$\sigma_1 - \sigma_2 = Y$
-Y
$\sigma_2 = -Y$

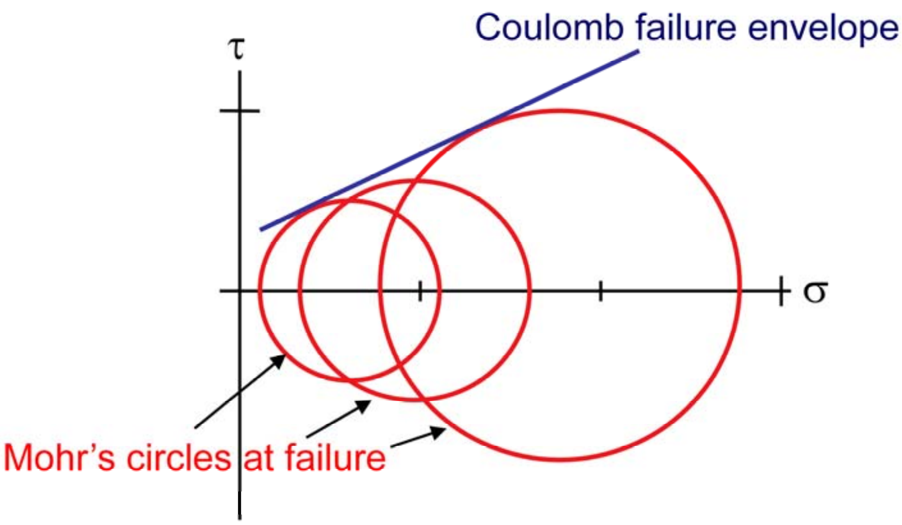
Mohr-Coulomb Failure Theory

Mohr's circle



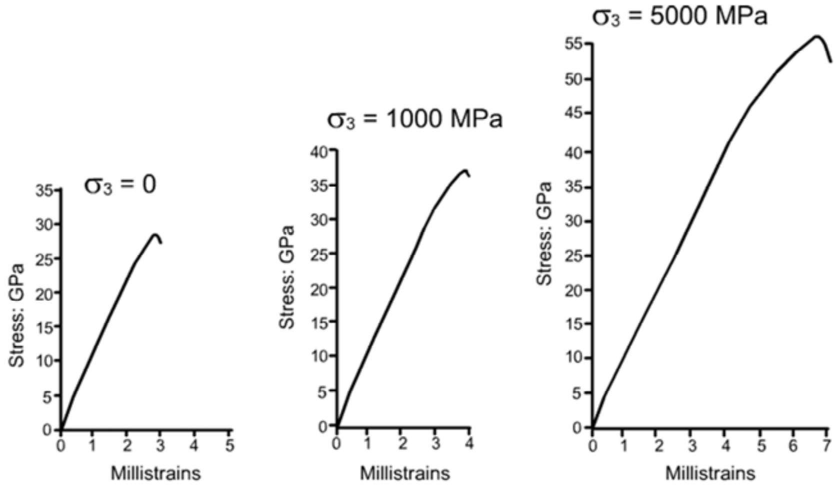
The Mohr's circle is a graphical representation of the stress state on an element in a body. Considering an element in plane stress (Fig. a) with σ_x, σ_y and τ_{xy} as the known normal and shear stresses, the Mohr's circle (Fig. b) can be used to find the normal and shear stresses on any plane defined by angle θ.

Mohr-Coulomb Failure Theory



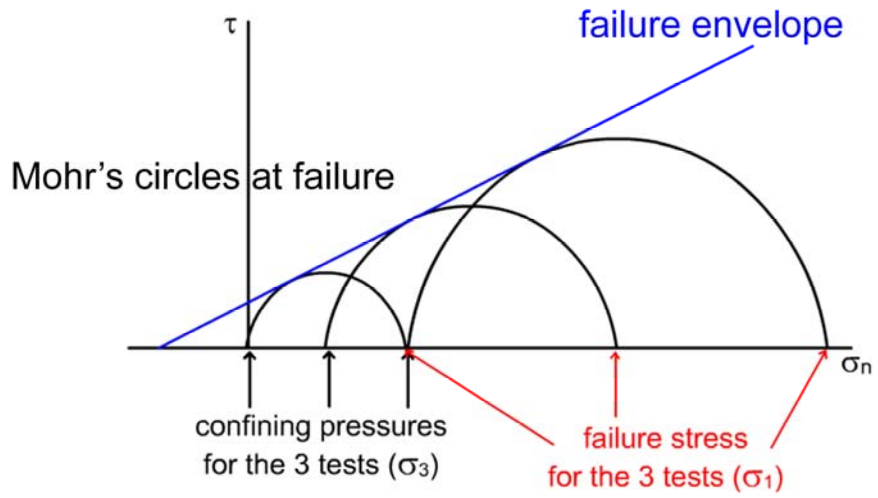
Failure is assumed to occur when the largest Mohr's circle representing the state of stress at a given point becomes tangent to (or exceeds) the failure envelope.

Mohr-Coulomb Failure Theory



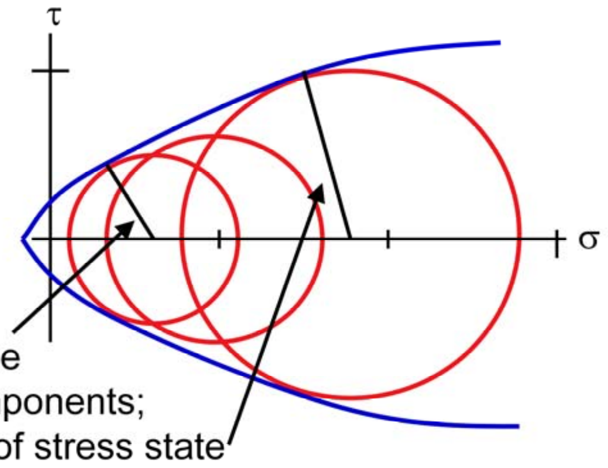
Construction of the failure envelope from test data; for example, results of tests of the same material under different confinement pressures and increasing axial stress.

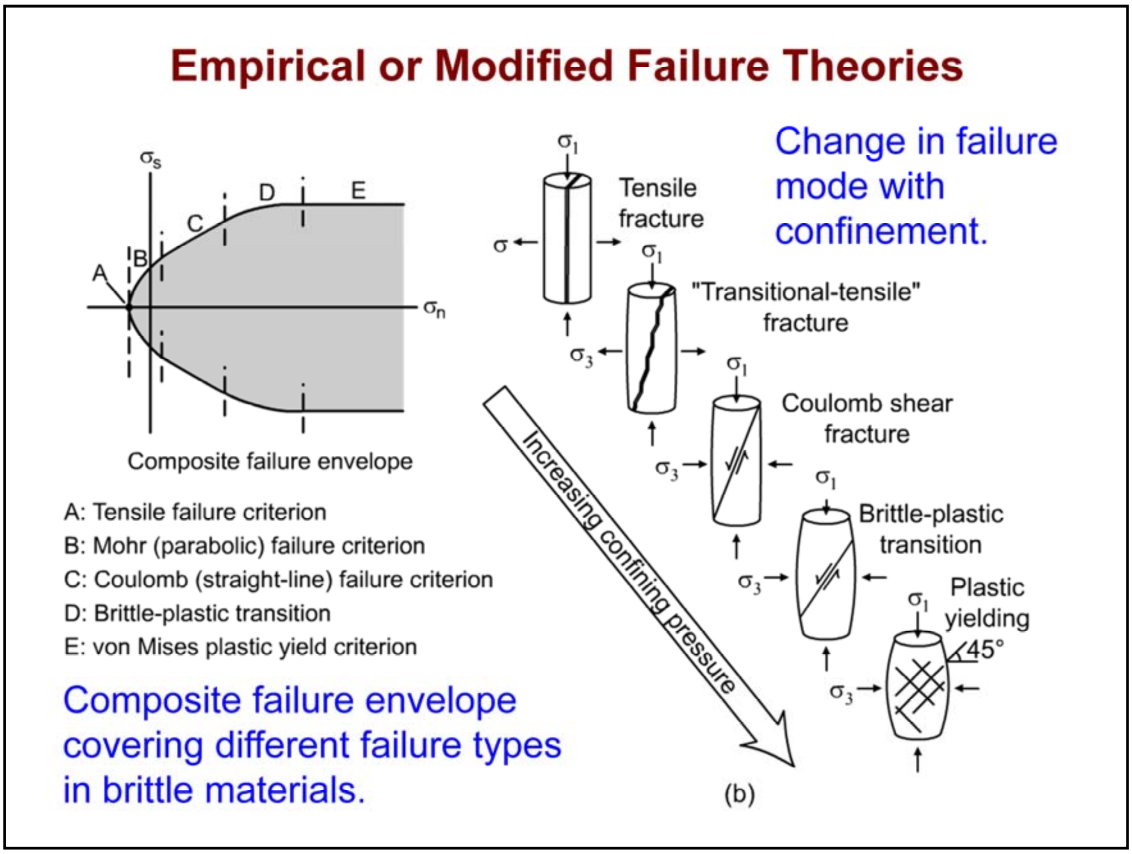
Mohr-Coulomb Failure Theory



Mohr-Coulomb Failure Theory

Coulomb originally defined the failure envelope as a straight line but Mohr showed that the slope decreases as the confining pressure increases (i.e., parabolic failure envelope is more general).




Empirical or Modified Failure Theories
σ_s
A
B
C
D
E
σ_n
Composite failure envelope
A: Tensile failure criterion
B: Mohr (parabolic) failure criterion
C: Coulomb (straight-line) failure criterion
D: Brittle-plastic transition
E: von Mises plastic yield criterion
Composite failure envelope covering different failure types in brittle materials.
Change in failure mode with confinement.
σ_1
σ
Tensile fracture
"Transitional-tensile" fracture
σ_3
Coulomb shear fracture
Brittle-plastic transition
Plastic yielding
45°
Increasing confining pressure
(b)

References

- *The Science and Technology of Civil Engineering Materials*, J.F. Young, S. Mindess, R.J. Gray & A. Bentur, Prentice Hall, 1998